Frequency Response Analysis

Bode plots · gain and phase margins · Nyquist stability

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MEC524 Control Systems · Handbook Chapter 5 · Syllabus topic 5 · Weeks 13–14

Semester 20264

How to use this chapter

Three documents, three different jobs. Use the right one.

These slides	Where every result *comes from*, and what it means. Follow the derivations here, step by step.
The handbook	The reference. Look up a rule, a table, a worked example. Chapter 5, § 5.0–5.4.
The tutorial	Where you find out whether you actually understood it. § 5.5.

Week 13 has no live lecture

Work through Topics A–C (Bode) on your own. Every topic ends with a *Check yourself* frame: answer it on paper *before* you turn the page. Post what did not make sense on the uFuture forum — it is answered in the **Week 14 live clinic**, which then continues with Nyquist (D–E).

Where Chapter 5 is going

1. **A. What frequency response is** — a sine in, a scaled and delayed sine out. Handbook § 5.0
2. **B. Bode plots by hand** — standard form, straight lines, corrections. Handbook § 5.1
3. **C. How close to instability?** — gain margin and phase margin. Handbook § 5.2
4. **D. The Nyquist plot** — one curve, and the count $Z = N + P$. Handbook § 5.3
5. **E. Margins on the Nyquist plane** — and what closeness to -1 means. Handbook § 5.4

Why this matters for PBL Milestone 2

You will tune a PID controller in the time domain, then *validate* it in the frequency domain. The numbers you report are the margins of this chapter.

You already know what frequency response is

Three things you have felt, all saying the same thing.

- **A car on a corrugated road.** Slowly, the body rides every bump. Fast, the wheels hammer and the cabin barely moves.
- **A speaker.** The same amplifier pushes every note, yet the small speaker gives thin bass: its output depends on the *frequency* of what you feed it.
- **Waving a long ruler.** Wave it slowly and the tip follows your hand. Wave it fast and the tip lags behind, or barely moves.

The one idea of Chapter 5

Push a system with a sine. Once the start-up has died away, it answers with a sine of the **same frequency** — only **bigger or smaller**, and **later**. Those two numbers, measured at every frequency, are its frequency response.

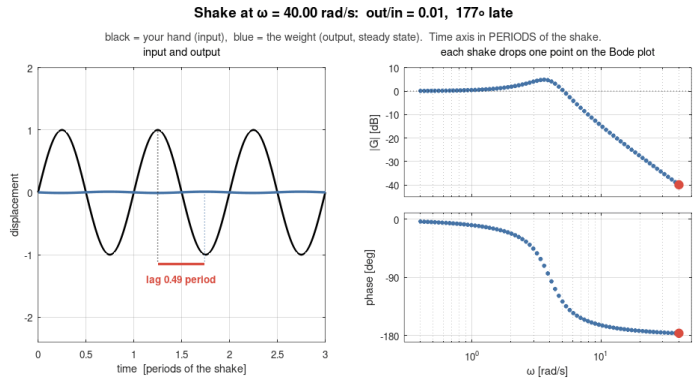
Shake a weight on a rubber band: slowly, just right, fast

Hold a rubber band with a weight on it, and move your hand up and down.

Your hand	**The weight moves…**	**…and arrives**
slowly	as much as your hand	with your hand, in step
at the bounce frequency	*more* than your hand	a quarter of a cycle late
fast	hardly at all	half a cycle late — going down as your hand goes up

- **How much**: the *amplitude ratio*, out/in.
- **How late**: the *phase lag*, in degrees — a full cycle late is 360°, so a quarter is 90° and a half is 180°.

Watch a frequency sweep



The weight on the band, $G(s) = 16/(s^2 + 2.4s + 16)$ ($\omega_n = 4$ rad/s, $\zeta = 0.3$). The hand's frequency rises from 0.4 to 40 rad/s; each steady shake drops one point on the Bode plot. A Bode plot is nothing more than this list of shakes. GIF: `ch5-anim/MEC524-ch5-sweep.gif`.

The sweep in numbers

Three of the shakes, read off $G(j\omega)$ (§5.0: $|G(j\omega)|$ is the ratio, $\angle G(j\omega)$ the phase):

ω [rad/s]	ratio $\lvert G\rvert$	in dB	lag	period	lag as a time
1	1.05	0.45	9.1°	6.28 s	0.159 s
$4 = \omega_n$	1.67	4.44	90.0°	1.57 s	0.393 s
$8 = 2\omega_n$	0.31	-10.2	158.2°	0.785 s	0.345 s

- dB is $20\log_{10}$(ratio): bigger than the input is *positive* dB, smaller is negative.
- At ω_n the ratio is exactly $1/(2\zeta) = 1.67$ and the lag exactly 90°, whatever ω_n is.
- The biggest ratio, 1.75, is at $\omega = 3.62$ rad/s — a little below ω_n.

Three things the sweep teaches — and beginners miss

1. **The frequency never changes.** Only size and timing do. If a linear system's output has a new frequency in it, you are still looking at the start-up transient, not the steady state.
2. **"Smaller and later" is the usual case, not a law.** Near a resonance the output is *bigger* than the input (the ω_n row: 1.67 times).
3. **Phase is a fraction of a period, not a fixed delay.** 90° at 4 rad/s is 0.393 s; 158.2° at 8 rad/s is only 0.345 s. More degrees can mean *less* time.

From here on

Topic A proves why the output is a pure sine of the same frequency; Topic B draws the whole sweep by hand, with straight lines.

A servo asked to follow a sine wave

A camera gimbal is commanded to sweep back and forth: $\theta_{\text{ref}}(t) = \sin \omega t$.

Slow sweep

- the gimbal follows almost exactly;
- same size, hardly any delay.

Fast sweep

- the gimbal swings *less* than commanded;
- and it arrives *late*.

Two numbers describe it at each frequency: **how much smaller** (amplitude ratio) and **how much later** (phase lag).

The engineering question of Chapter 5

From how the open loop answers sinusoids — which can be *measured* with a signal generator, no model needed — predict whether closing the loop is safe, and by how much.

Why the steady output is a sine of the same frequency

Input $u(t) = \sin \omega t$, so $U(s) = \dfrac{\omega}{s^2 + \omega^2}$. Output:

$$Y(s) = G(s)\,\frac{\omega}{(s - j\omega)(s + j\omega)} = \underbrace{\sum \frac{c_i}{s - p_i}}_{\text{poles of } G} + \frac{a}{s - j\omega} + \frac{\bar{a}}{s + j\omega}$$

Step 1 — the transient dies. G is stable, so every p_i has a negative real part and $c_i e^{p_i t} \to 0$.

Step 2 — the residue at $s = j\omega$.

$$a = \Big[(s - j\omega)\,Y(s)\Big]_{s=j\omega} = G(j\omega)\,\frac{\omega}{2j\omega} = \frac{G(j\omega)}{2j}$$

Step 3 — write $G(j\omega) = |G| e^{j\phi}$ and add the pair.

$$y_{ss}(t) = |G|\,\frac{e^{j(\omega t + \phi)} - e^{-j(\omega t + \phi)}}{2j} = \boxed{|G(j\omega)|\,\sin\big(\omega t + \angle G(j\omega)\big)}$$

Handbook § 5.0.1

Magnitudes multiply, decibels add

$G(j\omega)$ is one complex number per frequency:

$$G(j\omega) = \mathrm{Re} + j\,\mathrm{Im} = |G|\,e^{j\angle G}, \qquad |G| = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2}$$

For a product of factors, $|G_1 G_2| = |G_1|\,|G_2|$ and $\angle G_1 G_2 = \angle G_1 + \angle G_2$. Take logs and the product becomes a sum:

$$20\log_{10}|G_1 G_2| = 20\log_{10}|G_1| + 20\log_{10}|G_2|$$

$\lvert G\rvert$	0.01	0.1	0.5	0.707	1	1.414	2	10	100
dB	-40	-20	-6	-3	0	$+3$	$+6$	$+20$	$+40$

20 log, never 10 log

$|G|$ is an *amplitude* ratio. 10 log is for power ratios.

Handbook § 5.0.3–5.0.4

Worked example — first-order lag (1 of 2)

A thermocouple probe behaves as $G(s) = \dfrac{1}{\tau s + 1}$ with $\tau = 0.1$ s. Find $|G|$ and $\angle G$ at $\omega = 1$, 10, 100 rad/s.

Step 1 — substitute $s = j\omega$

$$G(j\omega) = \frac{1}{1 + j\,\omega\tau}$$

Step 2 — magnitude and phase of a quotient

$$|G| = \frac{1}{\sqrt{1 + (\omega\tau)^2}}, \qquad \angle G = 0 - \tan^{-1}(\omega\tau)$$

Step 3 — at $\omega = 10$ rad/s, $\omega\tau = 1$

$$|G| = \frac{1}{\sqrt{2}} = 0.707 = -3.01 \text{ dB}, \qquad \angle G = -\tan^{-1} 1 = -45^\circ$$

Handbook § 5.0.5

Worked example — first-order lag (2 of 2)

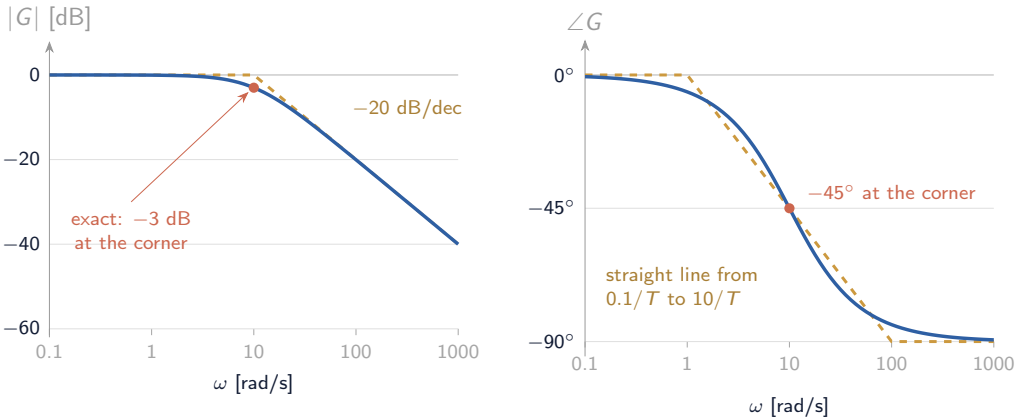
ω [rad/s]	$\omega\tau$	$\lvert G\rvert$ [dB]	$\angle G$ [deg]
1	0.1	-0.04	-5.7
10	1	-3.01	-45.0
100	10	-20.04	-84.3

- Below the **corner** $1/\tau = 10$ rad/s: $|G| \approx 0$ dB, $\angle G \approx 0^\circ$.
- *At* the corner: -3 dB and exactly -45°.
- Above it: -20 dB per decade, phase heading for -90°.

What -45° means in seconds

At $\omega = 10$ rad/s, $45^\circ = 0.785$ rad, so the output lags by $0.785/10 = 0.079$ s. Simulating $0.1\dot{y} + y = \sin 10t$ settles to amplitude 0.707 lagging 45° — the ODE and $G(j\omega)$ agree.

First-order lag — exact curve against the straight lines



— exact - - asymptote. The straight lines are never more than 3 dB or 5.7° out.

Check yourself A

A pressure sensor has $G(s) = \dfrac{20}{s+4}$.

1. Write G in the form $K/(1+Ts)$. State K in dB and the corner frequency.
2. Find $|G|$ in dB and $\angle G$ at $\omega = 4$ and $\omega = 40$ rad/s.
3. A 1-unit sine at 40 rad/s goes in. What amplitude comes out?

Work it on paper. The answers are on the next frame.

Check yourself A — answers

1. $G = \dfrac{5}{1 + s/4}$: $K = 5 = 13.98$ dB, corner 4 rad/s.
2. At $\omega = 4$ (the corner): $13.98 - 3.01 = 10.97$ dB, $-45°$.
   At $\omega = 40$ (a decade above): -6.06 dB, $-84.3°$.
3. $|G(j40)| = 5/\sqrt{101} = 0.498$ — about half the input.

If you got 6.99 instead of 13.98

You used $10 \log$. And if the straight line gave you $13.98 - 20 = -6.02$ dB at $\omega = 40$: good — the exact -6.06 dB is only 0.04 dB away.

Standard form — the step everyone skips

Bode axes: $\log\omega$ across; $20\log|G|$ in dB, and $\angle G$ in degrees, down. A **decade** is a factor of 10 in ω; an **octave** a factor of 2 (-20 dB/dec $= -6$ dB/octave).

$$G(s) = \frac{K\prod(1+T_i s)}{s^N\prod(1+T_j s)\prod\left(1+\frac{2\zeta}{\omega_n}s+\frac{s^2}{\omega_n^2}\right)}$$

Why this form: every bracket is 1 (0 dB) at low frequency, so K alone sets the level, and every corner sits at $1/T$ or ω_n.

The classic slip

$\dfrac{40}{s(s+2)(s+5)} = \dfrac{4}{s\,(1+s/2)(1+s/5)}$. The Bode gain is $K = 4$ (12.04 dB), *not* 40.

Handbook § 5.1.1–5.1.2

Factors 1 and 2: the gain and the integrator

Constant K. $20\log_{10} K$ dB at every frequency; phase 0° (-180° if $K < 0$).

Integrator $1/s$. At $s = j\omega$:

$$\left|\frac{1}{j\omega}\right| = \frac{1}{\omega} \quad \Rightarrow \quad 20\log\frac{1}{\omega} = -20\log\omega$$

Plotted against $\log\omega$ that is a **straight line of slope -20 dB/decade**, through 0 dB at $\omega = 1$. Phase: $\angle(1/j) = -90^\circ$, at every frequency.

Type N

$1/s^N$: slope $-20N$ dB/dec, phase $-90N^\circ$. The low-frequency slope of a Bode plot tells you the system type at a glance.

Handbook § 5.1.3

Factor 3: a first-order pole

$$20\log\left|\frac{1}{1+j\omega T}\right| = -10\log\bigl(1+(\omega T)^2\bigr)$$

Magnitude

- $\omega T \ll 1$: $\to 0$ dB.
- $\omega T \gg 1$: $\to -20\log(\omega T)$, slope -20 dB/dec, meeting 0 dB at the **corner** $\omega = 1/T$.
- At $\omega T = 1$: $-10\log 2 = -3.01$ dB — the largest error of the straight lines.

Phase $= -\tan^{-1}(\omega T)$

- $-5.7°$ at $0.1/T$, $-45°$ at $1/T$, $-84.3°$ at $10/T$.
- So: a straight ramp from $0°$ to $-90°$ across the two decades, error $\leq 5.7°$.

A zero $(1 + j\omega T)$ is the mirror image

0 dB then +20 dB/dec; phase $0° \to +90°$.

Handbook § 5.1.3

Factor 4: a quadratic pole pair

$$G(j\omega) = \frac{1}{1 - (\omega/\omega_n)^2 + j\,2\zeta\,\omega/\omega_n}$$

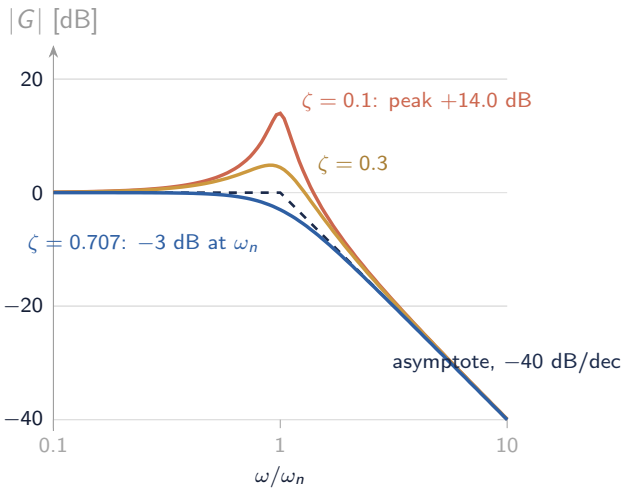
- $\omega \ll \omega_n$: $\to 0$ dB. $\quad \omega \gg \omega_n$: $\to -40\log(\omega/\omega_n)$, slope -40 dB/dec from ω_n.
- **At** $\omega = \omega_n$ the real part vanishes: $|G| = \dfrac{1}{2\zeta}$ and $\angle G = -90^\circ$. The straight line says 0 dB whatever ζ is.
- For $\zeta < 0.707$ the curve peaks:

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad \text{at} \quad \omega_r = \omega_n\sqrt{1-2\zeta^2}$$

- Phase: $0^\circ \to -180^\circ$, passing -90° at ω_n — more abruptly the smaller ζ is.

Handbook § 5.1.3

Quadratic factor — the peak the straight lines miss



At $\zeta = 0.1$ the straight line says 0 dB at ω_n; the truth is $20\log(1/0.2) = +14.0$ dB.

At $\zeta = 0.707$ there is no peak at all, and $|G(j\omega_n)| = -3$ dB.

Always correct a lightly damped quadratic by hand — no amount of sketching skill fixes a 14 dB error.

The sketching rule: add slopes left to right

1. Put $G(s)$ in **standard form**. List K, type N, and every corner frequency in increasing order.
2. Draw the **low-frequency line**: slope $-20N$ dB/dec, through $20\log K$ dB at $\omega = 1$.
3. Walk right. At each corner **change the slope**: -20 for a pole, $+20$ for a zero, ∓ 40 for a quadratic.
4. **Phase**: add the factor ramps; each passes $\mp 45^\circ$ at its own corner.
5. Where a number matters (a margin), **evaluate** $G(j\omega)$ **exactly** at that frequency. Never read it off the lines.

Handbook § 5.1.7

Worked example — asymptotic Bode (1 of 3)

Sketch the Bode plot of $G(s) = \dfrac{100}{s(s+10)(s+100)}$.

Step 1 — standard form

$$G(s) = \frac{100}{s \cdot 10(1+s/10) \cdot 100(1+s/100)} = \frac{0.1}{s\,(1+s/10)(1+s/100)}$$

$K = 0.1 = -20$ dB, type 1, corners $\omega_1 = 10$ and $\omega_2 = 100$ rad/s.

Step 2 — low-frequency line. $|G| \approx 0.1/\omega$: slope -20 dB/dec, -20 dB at $\omega = 1$.

Step 3 — slopes

Band	Slope [dB/dec]	Why
$\omega < 10$	-20	integrator
$10 < \omega < 100$	-40	+ pole at 10
$\omega > 100$	-60	+ pole at 100

Step 4 — walk the corners

- $\omega = 10$: one decade at -20 from -20 dB $\Rightarrow -40$ dB. Exact: -43.1 dB.
- $\omega = 100$: one decade at $-40 \Rightarrow -80$ dB. Exact: -83.1 dB.
- $\omega = 1000$: one decade at $-60 \Rightarrow -140$ dB.

Step 5 — phase, factor by factor

$$\angle G = -90^\circ - \tan^{-1}\frac{\omega}{10} - \tan^{-1}\frac{\omega}{100}$$

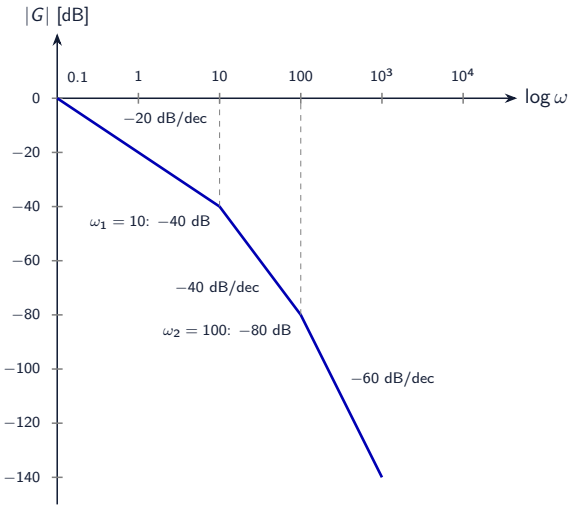
- $\omega = 10$: $-90 - 45 - 5.7 = -140.7^\circ$; $\omega = 100$: $-90 - 84.3 - 45 = -219.3^\circ$.

Step 6 — where is -180°? Need $\tan^{-1}\frac{\omega}{10} + \tan^{-1}\frac{\omega}{100} = 90^\circ$, i.e. the denominator of $\tan(A+B)$ vanishes:

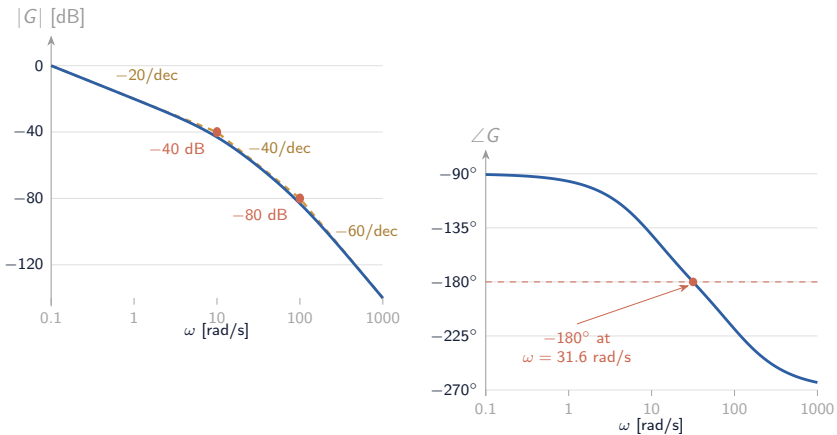
$$1 - \frac{\omega}{10}\cdot\frac{\omega}{100} = 0 \quad \Rightarrow \quad \omega = \sqrt{1000} = 31.6 \text{ rad/s}$$

Handbook § 5.1.4

The straight-line sketch, corner by corner



Worked example — asymptotic Bode (3 of 3)



— exact -- asymptote. Every break in slope is one corner of the standard form.

Check yourself B

$G(s) = \dfrac{20(s+1)}{s(s+10)}$ — this one has a **zero**.

1. Write the standard form. State K in dB, and which corner is a pole and which a zero.
2. Give the asymptotic slope in each band.
3. Asymptotic $|G|$ in dB at $\omega = 1$ and $\omega = 100$ rad/s.
4. Phase at very low and very high ω, and the exact phase at $\omega = \sqrt{10}$ rad/s.

Answers on the next frame.

Check yourself B — answers

1. $G = \dfrac{2\,(1+s)}{s\,(1+s/10)}$: $K = 2 = 6.02$ dB. Zero at 1, pole at 10 rad/s.
2. -20 ($\omega < 1$), $\quad 0$ ($1 < \omega < 10$), $\quad -20$ ($\omega > 10$) dB/dec.
3. $+6.02$ dB at $\omega = 1$, flat to $\omega = 10$, then -13.98 dB at $\omega = 100$ (exact: -14.02 dB).
4. -90° at both ends. In between the zero pulls it up: $-90 + \tan^{-1}\sqrt{10} - \tan^{-1}\frac{\sqrt{10}}{10} = -35.1^\circ$.

Why $\sqrt{10}$?

It is midway between the two corners on a log axis — where the zero's lead and the pole's lag are most out of balance.

Turning up the gain on a positioning servo

A pick-and-place arm runs under proportional control with gain K.

- Low K: sluggish, large following error.
- Raise K: faster — but it starts to overshoot and ring.
- Too high: it oscillates by itself and never settles.

Routh (Chapter 4) tells you the K at which it goes unstable. It does not tell you *how far away* you are today.

Two numbers answer that

Gain margin: how much more gain before it oscillates.
Phase margin: how much more delay (phase lag) before it oscillates.

Why $|G| = 1$ at -180° is the edge

Unity feedback: $T(s) = \dfrac{G(s)}{1 + G(s)}$. Closed-loop poles are the roots of $1 + G(s) = 0$.

Step 1. On the stability boundary a closed-loop pole sits on the imaginary axis, at $s = j\omega_0$:

$$1 + G(j\omega_0) = 0 \quad \Rightarrow \quad G(j\omega_0) = -1 = 1 \cdot e^{-j180^\circ}$$

Step 2. So *both* must hold at the same frequency: $|G(j\omega_0)| = 1$ (0 dB) **and** $\angle G(j\omega_0) = -180^\circ$.

Step 3 — two ways to miss the edge.

- At ω_{pc}, where the phase *is* -180°: how much can $|G|$ grow before it reaches 1? $\Rightarrow$ **GM**.
- At ω_{gc}, where $|G|$ *is* 1: how much more lag before -180°? $\Rightarrow$ **PM**.

Assumption

Open loop stable, no RHP poles. Otherwise use Nyquist (Topic D).

Gain margin and phase margin

Definitions

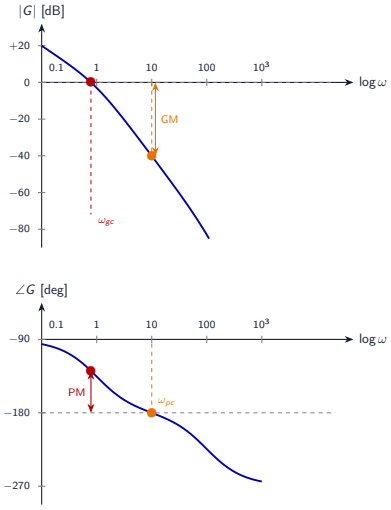
- **Gain crossover** ω_{gc}: $|G(j\omega_{gc})| = 0$ dB. **Phase crossover** ω_{pc}: $\angle G(j\omega_{pc}) = -180^\circ$.
- $\mathrm{PM} = 180^\circ + \angle G(j\omega_{gc})$
- $\mathrm{GM} = -20 \log_{10} |G(j\omega_{pc})|$ dB $= 1/|G(j\omega_{pc})|$ as a ratio

- Both positive $\Rightarrow$ closed loop stable. Either zero $\Rightarrow$ on the edge. Either negative $\Rightarrow$ unstable.
- Design targets: $\mathrm{PM} = 45$–60° for a well-damped loop ($\geq 30^\circ$ at worst); $\mathrm{GM} \geq 6$ dB.

GM is read on the *magnitude* plot at ω_{pc}. PM is read on the *phase* plot at ω_{gc}. Each is found on the *other* plot.

Handbook § 5.2.1–5.2.3

Reading both margins off one Bode plot



Worked example — margins of a third-order loop (1 of 3)

Unity feedback with $G(s) = \dfrac{40}{s(s+2)(s+5)}$. Find ω_{gc}, ω_{pc}, PM and GM.

Step 1 — standard form. $G = \dfrac{4}{s\,(1+s/2)(1+s/5)}$: $K = 4 = 12.04$ dB, type 1, corners 2 and 5 rad/s.

Step 2 — asymptotic magnitude.

- $+12.04$ dB at $\omega = 1$, falling at -20 dB/dec $\Rightarrow +6.02$ dB at $\omega = 2$.
- -40 dB/dec to $\omega = 5$: $6.02 - 40\log_{10} 2.5 = -9.9$ dB.

Step 3 — a first guess at ω_{gc}. Between the corners the straight line is $\dfrac{4}{\omega} \cdot \dfrac{2}{\omega} = \dfrac{8}{\omega^2}$, which equals 1 at

$$\omega = \sqrt{8} = 2.83 \text{ rad/s}$$

A first guess, not the answer: both corners are close to 2.83 — exactly where the straight lines are least trustworthy.

Worked example — margins of a third-order loop (2 of 3)

Step 4 — phase crossover, exactly.

$$-90^\circ - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{5} = -180^\circ \;\Rightarrow\; 1 - \frac{\omega}{2}\cdot\frac{\omega}{5} = 0 \;\Rightarrow\; \omega_{pc} = \sqrt{10} = 3.16 \text{ rad/s}$$

Step 5 — magnitude there.

$$|G(j\omega_{pc})| = \frac{40}{\sqrt{10}\,\sqrt{2^2+10}\,\sqrt{5^2+10}} = \frac{40}{3.16 \times 3.74 \times 5.92} = \frac{40}{70} = 0.571$$

Step 6 — gain margin.

$$\mathrm{GM} = \frac{1}{0.571} = 1.75 \qquad \Rightarrow \qquad 20\log_{10} 1.75 = \boxed{4.86 \text{ dB}}$$

Cross-check with Routh

$s^3 + 7s^2 + 10s + K = 0$ goes unstable at $K_{cr} = 7 \times 10 = 70$, with roots at $\pm j\sqrt{10}$. And $70/40 = 1.75$ — the same margin, the same frequency.

Worked example — margins of a third-order loop (3 of 3)

Step 7 — gain crossover, exactly. Square $|G(j\omega)| = 1$:

$$\omega^2(\omega^2+4)(\omega^2+25) = 1600 \quad \Rightarrow \quad \omega_{gc} = 2.35 \text{ rad/s}$$

(a cubic in ω^2 — solve by a few trials or bisection).
Step 8 — phase margin.

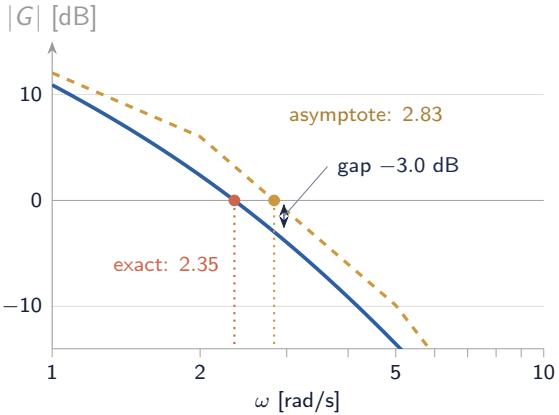
$$\angle G(j2.35) = -164.7^\circ \qquad \Rightarrow \qquad \text{PM} = 180 - 164.7 = \boxed{15.3^\circ}$$

Conclusion. Both margins positive: stable. But 15.3° is far below 45° — this loop will ring badly.

What the first guess would have given

At $\omega = 2.83$: $\angle G = -90 - 54.7 - 29.5 = -174.2^\circ$, "PM" $= 5.8^\circ$. Wrong by a factor of 2.6, because at 2.83 rad/s the true $|G|$ is -3.0 dB, not 0 dB. The straight line locates the crossover; only $|G(j\omega)| = 1$ solved exactly gives the phase margin, 15.3°.

Where the straight line misleads



$G = \dfrac{40}{s(s+2)(s+5)}$, zoomed to $1 \le \omega \le 10$ rad/s.

The asymptote crosses 0 dB at 2.83 rad/s; the exact curve at 2.35. The gap is the corner correction.

Rule: sketch with straight lines, then solve $|G(j\omega)| = 1$ exactly before you quote a phase margin.

Phase margin is damping in disguise

For $G(s) = \dfrac{\omega_n^2}{s(s + 2\zeta\omega_n)}$ the closed loop is exactly the second-order system of Chapter 4, and solving $|G| = 1$ gives

$$\mathrm{PM} = \tan^{-1} \frac{2\zeta}{\sqrt{\sqrt{1 + 4\zeta^4} - 2\zeta^2}} \approx 100\,\zeta \text{ degrees} \quad (\zeta \lesssim 0.6)$$

ζ	0.2	0.4	0.5	0.6	0.7
PM [deg]	22.6	43.1	51.8	59.2	65.2

So a PM target is a damping target: $\mathrm{PM} \approx 50^\circ$ means $\zeta \approx 0.5$, which means about 16.3% overshoot.

Sanity check on the worked example

$\mathrm{PM} = 15.3^\circ$ suggests $\zeta \approx 0.15$. The actual dominant closed-loop poles of $s^3 + 7s^2 + 10s + 40$ have $\zeta = 0.12$. Right ballpark — for a third-order loop the rule is a guide, not a formula.

Handbook § 5.2.3

What the gain knob does to both margins

- Raising K lifts the whole magnitude curve by $20\log(K_{\text{new}}/K_{\text{old}})$ — doubling K adds 6.02 dB. **The phase curve does not move.**
- So ω_{gc} slides right, to where the phase is lower: **PM falls**. And GM falls by exactly the dB you added.
- Worked example: K from 40 to 70 adds 4.86 dB, GM becomes 0 dB — the Routh limit.

Lead compensator

adds phase near ω_{gc}: PM up *without* cutting K.

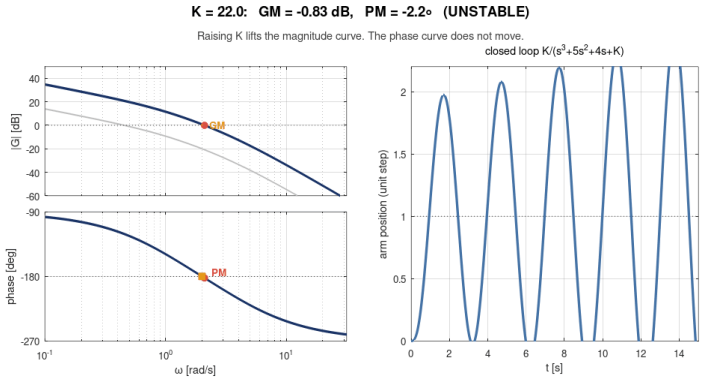
Lag compensator

cuts mid-band gain: ω_{gc} moves left, PM up, response slower.

PBL Milestone 2: the derivative action of your PID is a lead. Check its effect on PM.

Handbook § 5.2.5

Watch the gain knob



A pick-and-place arm, $G(s) = K/[s(s+1)(s+4)]$, with K turned from 2 to 22. Left: the magnitude curve rises (grey is $K = 2$), the phase curve stays put, PM and GM shrink. Right: the closed-loop step response rings longer, then never settles at $K = 20$, then grows. GIF: `ch5-anim/MEC524-ch5-gain.gif`.

The gain knob in numbers

Same arm, $G(s) = \dfrac{K}{s(s+1)(s+4)}$. Phase crossover: $\angle G = -180^\circ$ at $\omega_{pc} = 2$ rad/s for *every* K.

K	GM [dB]	ω_{gc} [rad/s]	PM	step overshoot
2	20.0	0.453	59.2°	8.7 %
5	12.0	0.904	35.1°	35.3 %
10	6.02	1.38	16.8°	62.0 %
20	0	2.00	0°	never settles

- GM falls by exactly the dB added: K doubled from 5 to 10 costs 6.02 dB. PM falls faster than any rule of thumb.
- **Cross-check with Routh** (Chapter 4): $s^3 + 5s^2 + 4s + K$ is on the edge when $5 \times 4 = K$, i.e. $K = 20$, oscillating at $\omega = \sqrt{4} = 2$ rad/s — the same ω_{pc}.

Check yourself C

Unity feedback with $G(s) = \dfrac{10}{s(s+2)}$.

1. Find the exact gain crossover frequency ω_{gc}.
2. Find the phase margin.
3. Find the gain margin. (Careful.)
4. Find ζ of the closed loop and compare PM with 100ζ.

Answers on the next frame.

Check yourself C — answers

1. $\dfrac{100}{\omega^2(\omega^2+4)} = 1 \Rightarrow \omega^4 + 4\omega^2 - 100 = 0 \Rightarrow \omega^2 = -2 + \sqrt{104} = 8.20, \quad \omega_{gc} = 2.86$ rad/s.

2. $\angle G = -90^\circ - \tan^{-1}(2.86/2) = -145.1^\circ \Rightarrow \mathrm{PM} = 34.9^\circ$.

3. The phase only *approaches* -180° as $\omega \to \infty$: there is no ω_{pc}, so $\mathrm{GM} = \infty$. No gain $K > 0$ makes this loop unstable.

4. Closed loop $s^2 + 2s + 10$: $\omega_n = \sqrt{10}$, $\zeta = 2/(2\sqrt{10}) = 0.316$. $\quad 100\zeta = 31.6$ vs. exact 34.9° — close.

"GM = 0" is not the same as "GM = ∞"

No phase crossover means infinite gain margin, not zero. Say which, and why.

When the open loop is itself unstable

A magnetic levitation rig, a self-balancing robot, a modern fighter aircraft: **without feedback they fall over**. The open loop has a pole in the right half-plane.

- Topic C assumed a stable open loop. With an RHP pole, "both margins positive" no longer means stable — and a loop can be stable with a margin that looks negative.
- We need a test that *counts* closed-loop RHP poles, from open-loop frequency response alone.

The Nyquist plot

Plot $G(j\omega)$ as one curve in the complex plane — $\mathrm{Re}\, G$ across, $\mathrm{Im}\, G$ up — as ω runs from $-\infty$ to ∞. The $\omega < 0$ half is the mirror image of the $\omega > 0$ half about the real axis.

Handbook § 5.3.1

Worked example — first-order plant on the Nyquist plane

$G(s) = \dfrac{K}{s+a}$, with $K, a > 0$.

Step 1 — real and imaginary parts. Multiply by the conjugate:

$$G(j\omega) = \frac{K(a - j\omega)}{a^2 + \omega^2} : \qquad \mathrm{Re} = \frac{Ka}{a^2+\omega^2}, \quad \mathrm{Im} = \frac{-K\omega}{a^2+\omega^2}$$

Step 2 — key points. $\omega = 0$: K/a. $\omega = a$: $\left(\frac{K}{2a}, -\frac{K}{2a}\right)$. $\omega \to \infty$: the origin.

Step 3 — the shape. $\mathrm{Re}^2 + \mathrm{Im}^2 = \dfrac{K^2}{a^2+\omega^2} = \dfrac{K}{a}\,\mathrm{Re}$, which rearranges to

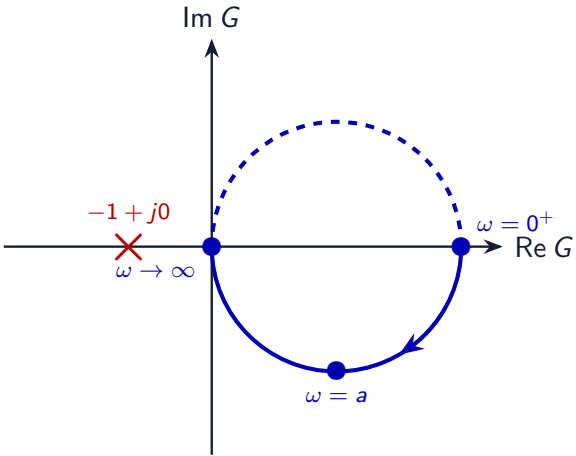
$$\left(\mathrm{Re} - \frac{K}{2a}\right)^2 + \mathrm{Im}^2 = \left(\frac{K}{2a}\right)^2$$

a circle through the origin and K/a.

Step 4 — stability. $\mathrm{Re}\, G > 0$ always, so the curve never gets near -1: stable for every $K > 0$.

Handbook § 5.3.4

The first-order Nyquist plot



Why -1, and what $Z = N + P$ counts

Step 1. Let $F(s) = 1 + G(s) = \dfrac{\text{den} + \text{num}}{\text{den}}$.

- zeros of F = closed-loop poles; poles of F = open-loop poles.

Step 2 — the argument principle. Take s once *clockwise* round a closed contour. $F(s)$ encircles the origin clockwise

$$N = Z - P \quad \text{times} \qquad (Z \text{ zeros},\ P \text{ poles of } F \text{ inside})$$

Step 3. Choose the contour to enclose the whole right half-plane. Then Z = closed-loop RHP poles, P = open-loop RHP poles.

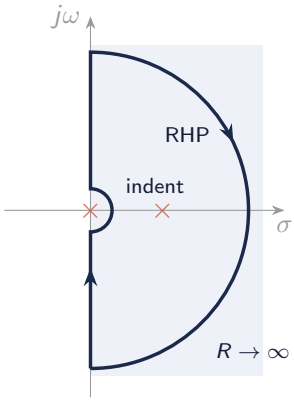
Step 4. F encircling 0 is the same as $G = F - 1$ encircling -1. That is why -1 is the critical point.

The Nyquist criterion

$$Z = N + P, \qquad \text{stable} \iff Z = 0 \iff N = -P$$

N counts clockwise encirclements of -1 as positive, counter-clockwise as negative.

The Nyquist contour, and poles on the axis



- Up the $j\omega$ axis, round a semicircle of radius $R \to \infty$: the whole right half-plane, clockwise.
- A pole *on* the axis (an integrator) is stepped round by a tiny semicircle to its right, so it is *not* counted in P.
- That indent maps to an arc of **infinite radius**, turning 180° clockwise per integrator. Draw it — it closes the curve.
- For a strictly proper G, the big semicircle maps to the origin.

Handbook §5.3.2–5.3.3

Worked example — the integrator plant $K/[s(s+a)]$

Step 1 — parts. $G(j\omega) = \dfrac{K}{j\omega(a+j\omega)}$: $\quad \mathrm{Re} = \dfrac{-K}{a^2+\omega^2}, \quad \mathrm{Im} = \dfrac{-Ka}{\omega(a^2+\omega^2)}$.

Step 2 — ends. $\omega \to 0^+$: $\mathrm{Re} \to -K/a^2$, $\mathrm{Im} \to -\infty$ (a vertical asymptote). $\omega \to \infty$: into the origin from the third quadrant.

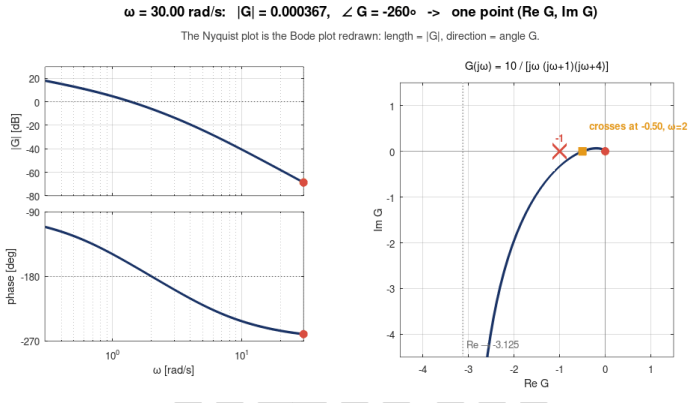
Step 3 — crossings. $\mathrm{Im} < 0$ for every $\omega > 0$: the curve never crosses the negative real axis, however large K is.

Step 4 — count. $P = 0$ (the pole at $s = 0$ is indented round). -1 is never encircled, so $N = 0$ and $Z = 0$: **stable for every** $K > 0$.

Cross-check. $s^2 + as + K$: all coefficients positive for any $K > 0$. ✓

A large K pushes the asymptote $\mathrm{Re} = -K/a^2$ left of -1, but the branches pass above and below -1 without crossing the axis there: still no encirclement.

Watch the Nyquist plot being drawn



The arm of Topic C at $K = 10$: one more pole than the integrator plant above. At each ω the Bode plot gives a length $|G|$ and a direction $\angle G$; the point at that length and direction is the Nyquist plot. The same red marker moves on both. GIF: `ch5-anim/MEC524-ch5-nyq.gif`.

Reading the curve you just watched

$$G(j\omega) = \frac{10}{j\omega(j\omega+1)(j\omega+4)}$$

- **One point.** At $\omega = 1$: $|G| = 1.715$, $\angle G = -149.0°$, so the point is $(-1.471, -0.882)$. Bode and Nyquist are the same numbers, drawn two ways.
- **Start.** $\omega \to 0^+$: $|G| \to \infty$ along the asymptote $\mathrm{Re}\, G \to -K(a+b)/(ab)^2 = -3.125$.
- **The crossing.** Unlike $K/[s(s+a)]$, the third pole bends the curve across the negative real axis: at $\omega = 2$ rad/s, at -0.50.
- **The verdict.** -0.50 is to the right of -1: no encirclement, $P = 0$, so $Z = 0$ — stable. The gain could double before the crossing reaches -1: $\mathrm{GM} = 1/0.50 = 2$, i.e. 6.02 dB, the same as the $K = 10$ row of the Bode table.

Worked example — an open-loop unstable plant (1 of 2)

Unity feedback with $G(s) = \dfrac{K}{(s+2)(s-1)}$. For which K is the closed loop stable?

Step 1 — count P. Poles at -2 and $+1$: $P = 1$. Stability needs $N = -1$, one *counter-clockwise* encirclement.

Step 2 — parts. $(2 + j\omega)(-1 + j\omega) = -(2 + \omega^2) + j\omega$, so

$$G(j\omega) = \frac{K\big[-(2+\omega^2) - j\omega\big]}{(2+\omega^2)^2 + \omega^2}$$

Step 3 — read the signs. For $\omega > 0$: $\mathrm{Re} < 0$ and $\mathrm{Im} < 0$ — the third quadrant. $G(j0) = -K/2$; $G \to 0$ as $\omega \to \infty$.

Step 4 — direction. The mirror branch ($\omega < 0$) runs from the origin across the *top* to $-K/2$; the $\omega > 0$ branch returns along the *bottom*. Left along the top, right along the bottom: **counter-clockwise**.

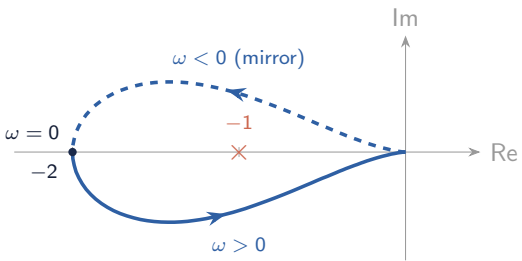
Worked example — an open-loop unstable plant (2 of 2)

Step 5 — where is -1?

- $K = 4$: the loop reaches -2, so -1 is inside. $N = -1$, $Z = N + P = 0$: **stable**.
- $K = 1$: the loop reaches -0.5; -1 is outside. $N = 0$, $Z = 1$: **unstable**.
- Boundary: $-K/2 = -1$. Stable for $\boxed{K > 2}$.

Cross-check. $s^2 + s + (K - 2)$: stable iff $K > 2$. ✓

Here feedback gain *creates* stability — more gain is safer, the opposite of Topic C.



$K = 4$. Solid: $\omega > 0$. Dashed: mirror.

Check yourself D

Unity feedback with $G(s) = \dfrac{K}{(s+1)^3}$, $K > 0$.

1. Find the frequency at which the Nyquist plot crosses the negative real axis, and the crossing point in terms of K.
2. Find the critical gain K_{cr}.
3. Give N and Z for $K = 4$ and for $K = 10$.
4. Check K_{cr} with Routh.

Answers on the next frame.

Check yourself D — answers

1. $-3\tan^{-1}\omega = -180^\circ \Rightarrow \omega = \tan 60^\circ = \sqrt{3} = 1.73$ rad/s. There $|G| = K/(1+3)^{3/2} = K/8$: the crossing is at $-K/8$.

2. $-K/8 = -1 \Rightarrow K_{cr} = 8$.

3. $P = 0$. $K = 4$: crossing at -0.5, $N = 0$, $Z = 0$, stable.
   $K = 10$: crossing at -1.25. Both branches pass left of -1: $N = 2$ clockwise, $Z = 2$ — two closed-loop RHP poles.

4. $s^3 + 3s^2 + 3s + (1+K)$: needs $3 \times 3 > 1 + K$, so $K < 8$. ✓

Why $N = 2$, not 1

The $\omega > 0$ branch and its mirror each cross the real axis at -1.25. Count the whole closed curve, not half of it.

Margins on the Nyquist plane

- **Gain margin.** The curve crosses the negative real axis at $-a$ (at ω_{pc}). Multiply the gain by $1/a$ and that crossing lands on -1:

$$\mathrm{GM} = \frac{1}{a} = -20 \log_{10} a \ \mathrm{dB}$$

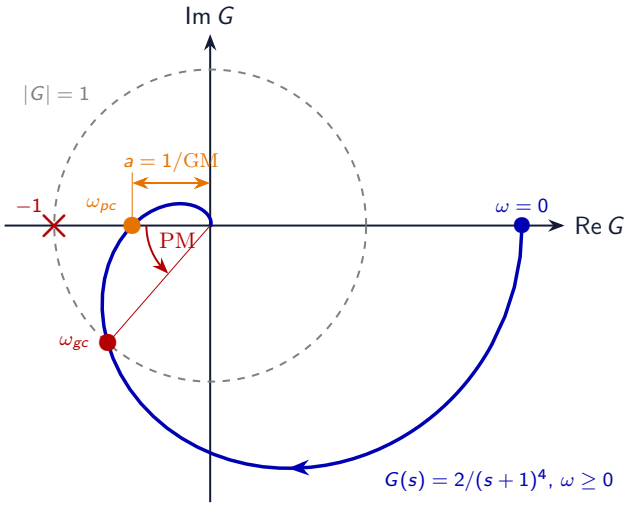
- **Phase margin.** The curve crosses the unit circle $|G| = 1$ at ω_{gc}. PM is the angle from the negative real axis to that point.
- **Vector margin.** The closest approach of the curve to -1, $\min_\omega |1 + G(j\omega)|$ — one number covering both.

Only for $P = 0$

With open-loop RHP poles, stability comes from the count $Z = N + P$. Do not read margins first.

Handbook § 5.4.1

Both margins on one Nyquist plot



Worked example — margins read on the Nyquist plane

Same loop as Topic C:
$G(s) = \dfrac{40}{s(s+2)(s+5)}$.

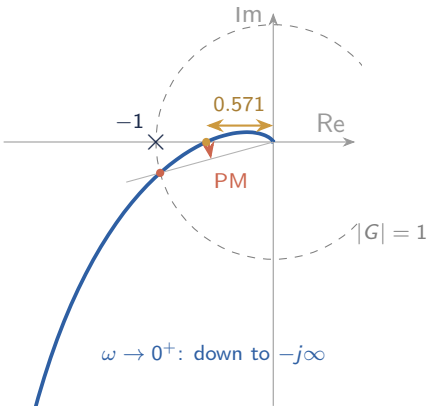
Negative real axis: crossed at $\omega_{pc} = 3.16$ rad/s, at -0.571.
$\mathrm{GM} = 1/0.571 = 1.75 = 4.86$ dB.

Unit circle: crossed at $\omega_{gc} = 2.35$ rad/s, at angle $-164.7°$. $\mathrm{PM} = 15.3°$.

Closest approach to -1: vector margin 0.23.

As $\omega \to 0^+$ the curve heads to $-j\infty$ along $\mathrm{Re} = -4(0.5 + 0.2) = -2.8$ (off the frame).

Same numbers as the Bode plot — it is the same information.



Worked example — integrator and RHP pole together (1 of 2)

$$G(s)H(s) = \frac{K(s+1)}{s(s-1)(s+6)}, \quad K > 0.$$

Step 1 — count. Pole at $+1$: $P = 1$. Pole at 0: indented round, not counted. Need $N = -1$.

Step 2 — parts. The denominator at $s = j\omega$ is $-5\omega^2 - j\omega(\omega^2 + 6)$. Multiply through by its conjugate:

$$G(j\omega) = \frac{K\big[-(\omega^4 + 11\omega^2) + j\,\omega(6 - 4\omega^2)\big]}{25\omega^4 + \omega^2(\omega^2 + 6)^2}$$

Step 3 — real-axis crossing. $\mathrm{Im} = 0$ at $\omega^2 = 1.5$, $\omega = 1.22$ rad/s. Substituting gives $\mathrm{Re} = -K/7.5$.

Step 4 — ends. $\omega \to 0^+$: $G \approx \dfrac{K}{j\omega(-1)(6)} \to +j\infty$. $\omega \to \infty$: $G \approx K/(j\omega)^2 \to 0$ from -180°.

Worked example — integrator and RHP pole together (2 of 2)

Step 5 — close the curve. The indent at $s = 0$ maps to a 180° clockwise arc at infinite radius: from $-j\infty$ ($\omega = 0^-$) through the left half-plane to $+j\infty$ ($\omega = 0^+$).

Step 6 — count for two gains.

K	crossing	-1 is…	N	$Z = N + P$	closed loop
10	-1.33	inside	-1	0	stable
5	-0.67	outside the loop	$+1$	2	unstable

Boundary $-K/7.5 = -1$: stable for $\boxed{K > 7.5}$.
Cross-check. $s^3 + 5s^2 + (K-6)s + K$: the s^1 row of Routh is $(4K - 30)/5$, positive iff $K > 7.5$. At $K = 7.5$ the roots sit at $\pm j1.22$. ✓

Margins would point the wrong way

Here raising K *stabilises* the loop. A Bode "gain margin" read in the usual way says the opposite. With $P \neq 0$: count.

Closeness to -1 is closed-loop resonance

$$T(j\omega) = \frac{G(j\omega)}{1 + G(j\omega)} \qquad \Rightarrow \qquad |T(j\omega)| = \frac{|G(j\omega)|}{|1 + G(j\omega)|}$$

$|1 + G(j\omega)|$ is the **distance** from the point $G(j\omega)$ to -1.

- Curve passes close to -1: that distance is small, $|T|$ peaks — a closed-loop *resonance* at that frequency.
- Curve stays far away: $|T|$ is smooth, no peak.

Three views of one design

For a dominant second-order pair with $\zeta = 0.5$: $\quad M_p = 16.3\%$ ·

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.155 \;\cdot\; \mathrm{PM} = 51.8^\circ.$$

Curves of constant $|T|$ are circles in the G-plane (M-circles): the largest one the plot touches is the resonance peak.

Handbook §5.4.4–5.4.6

Check yourself E

Take $G(s) = \dfrac{4}{(s+1)^3}$ from Check yourself D.

1. Gain margin, as a ratio and in dB.
2. Gain crossover frequency and phase margin.
3. Is the closed loop stable? Does it meet $\mathrm{GM} \geq 6$ dB and $\mathrm{PM} \geq 45^\circ$?

Answers on the next frame.

Check yourself E — answers

1. Crossing at $-4/8 = -0.5$: $\mathrm{GM} = 2 = 6.02$ dB.

2. $\dfrac{4}{(1+\omega^2)^{3/2}} = 1 \Rightarrow 1+\omega^2 = 4^{2/3} = 2.52, \quad \omega_{gc} = 1.23$ rad/s.
   $\angle G = -3\tan^{-1} 1.23 = -152.9^\circ \Rightarrow \mathrm{PM} = 27.1^\circ$.

3. Stable: $P = 0$ and $N = 0$ (both margins positive). GM just meets 6 dB; PM falls well short of 45° — expect a strongly oscillatory step response.

Margins are two separate tests

A loop can pass one and fail the other. Report both, each with its crossover frequency.

Three ways to lose marks here

1. **Reading ω_{gc} off the straight lines.**
   Near a corner the asymptote is up to 3 dB high. In the worked example that turned $\mathrm{PM} = 15.3^\circ$ into 5.8°. Solve $|G(j\omega)| = 1$ exactly.

2. **Swapping the crossovers — or the logs.**
   GM is read at ω_{pc} (phase -180°) on the magnitude plot; PM at ω_{gc} (0 dB) on the phase plot. Decibels are $20\log$, and K is the *standard-form* gain.

3. **Counting encirclements without P.**
   $Z = N + P$, clockwise positive. Count the whole closed curve — both branches and the indent arc. With $P \neq 0$, margins do not decide stability.

What the assessment will ask

Assessment	Week	Weight	What it asks from this chapter
Test 2	After Wk 14	30% CLO1+CLO2	Asymptotic Bode sketch; crossovers, gain and phase margins; Nyquist sketch and stability by $Z = N + P$.
PBL Milestone 2	Wk 14	10% CLO3	Frequency-domain validation of your PID loop: margins of $C(s)G(s)$ against the time response.

What a full-mark answer looks like

Standard form first, with K, type and corners · every slope and corner value labelled on the sketch · crossovers solved exactly, margins with sign and unit · a stability verdict cross-checked by Routh or the closed-loop roots.

What to take away

1. A stable LTI system answers $\sin \omega t$ with $|G(j\omega)| \sin(\omega t + \angle G(j\omega))$ — frequency response is $G(s)$ at $s = j\omega$.
2. Standard form first. Slopes add; each corner bends the line by ± 20 (or ± 40) dB/dec. Correct by hand near corners.
3. GM at ω_{pc}, PM at ω_{gc}; both positive means stable for a stable open loop. $\mathrm{PM} \approx 100\zeta$.
4. $Z = N + P$: the closed loop is stable when the plot encircles -1 exactly P times counter-clockwise.

Handbook § 5.0–5.4. Worked examples: § 5.1.4, § 5.2.4, § 5.3.4–5.3.6, § 5.4.2–5.4.3.

Tutorial: § 5.5 — attempt it before the Week 14 clinic.