Modelling of Systems

mechanical · electrical · electromechanical · fluid

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MEC524 Control Systems · Handbook Chapter 3 · Syllabus Chapter 3 · Weeks 6–7

Semester 20264

How to use this chapter

Three documents, three different jobs. Use the right one.

These slides	Where every model *comes from*: the free-body diagram, the law, the ODE, the transfer function. Follow the derivations here.
The handbook	The reference. Look up an elemental relation, a table, a worked example. Chapter 3, § 3.0–3.8.
The tutorial	Where you find out whether you can do it alone. § 3.8.

Where this sits

In Chapter 2 every ODE was given. This chapter derives them from the physics, and ends in the transfer functions and block diagrams you already know.

Where Chapter 3 is going

By the end of week 7 you should be able to take a real machine and produce the one thing every later chapter starts from: its transfer function.



1. **Week 6** — elements, free-body diagrams, Newton; circuits. Handbook § 3.0–3.3, 3.5
2. **Week 7** — the DC motor and gearboxes; tanks, and the energy method. Handbook § 3.7, 3.6, 3.4

Chapters 4 and 5 start from $G(s)$

A wrong model gives a wrong overshoot, gain and stability margin later — and nothing downstream will warn you.

Why a control engineer builds a model

A solar tracker: a DC motor turns a panel through a gearbox. Before anyone buys the amplifier, somebody must answer: Handbook § 3.0

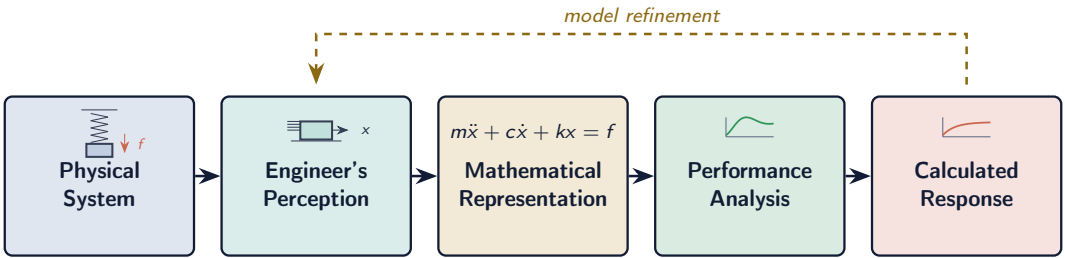
- How far does the panel turn for one volt?
- Does it overshoot the sun, or creep up on it?
- At what amplifier gain does the loop go unstable?

None of these can be read off the hardware on day one. All of them can be read off a *model*.

A model is chosen, not found

Keep what decides the answer (inertia, back-EMF, gear ratio). Drop what does not (paint, bolt stretch). When the model and the measurement disagree, go back and refine the choice, not the measurement.

The modelling loop, from hardware to response



One recipe, four domains

Handbook § 3.0, 3.5, 3.6

Domain	Law	Stores energy	Dissipates
Mechanical	$\sum F = m\ddot{x}$, $\sum M = I\ddot{\theta}$	mass, spring	damper
Electrical	KVL, KCL	inductor, capacitor	resistor
Fluid	storage = in − out	tank, long pipe	valve
Electromechanical	KVL + Newton + two couplings	all of the above	R_a, bearing

Then the same three moves every time: *element relations* → *law* → *Laplace with zero initial conditions*.

The order rule

The order of $G(s)$ equals the number of independent energy stores. Count them *before* you start: it tells you the degree of the denominator you must end up with.

Four ways to write the same dynamics

A mass m on a spring k and damper c, pushed by $f(t)$: Handbook § 3.1

Configuration	$m\ddot{x} + c\dot{x} + kx = f$	straight from the FBD
State space	$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} f$	simulation, MIMO
Input–output ODE	one ODE in input and output only	internal states eliminated
Transfer function	$\dfrac{X(s)}{F(s)} = \dfrac{1}{ms^2 + cs + k}$	Chapters 4–5

Same physics, four dialects. This chapter is how you earn the last one.

From ODE to transfer function in one line

Start. $m\ddot{x} + c\dot{x} + kx = f(t)$, every initial condition zero. Handbook § 3.1.5

Laplace. $\frac{d}{dt} \to s$, $\frac{d^2}{dt^2} \to s^2$: $\quad (ms^2 + cs + k)\,X(s) = F(s)$

Divide. $\boxed{G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}}$

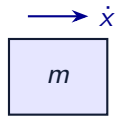
Read off. Poles from the denominator; $\omega_n = \sqrt{k/m}$, $\zeta = c/(2\sqrt{km})$, static gain $G(0) = 1/k$.

Zero initial conditions are part of the definition

$G(s)$ describes the *system*, not one particular run of it. Initial conditions belong to the time response (Chapter 4).

Three elements, three jobs

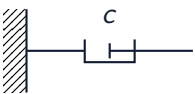
Every lumped mechanical model is built from three ideal elements. Handbook § 3.2



mass
stores $\frac{1}{2}m\dot{x}^2$
(kinetic)



spring
stores $\frac{1}{2}kx^2$
(potential)



damper
dissipates
(heat)

A car suspension has all three, for a reason

Remove the damper and the car bounces forever. Remove the spring and nothing carries the static load. Each element does one job no other can.

Elemental relations — and their Laplace form

Handbook § 3.2.2–3.2.4

Element	Translational	Rotational	Laplace, zero ICs
mass / inertia	$F = m\ddot{x}$	$T = J\ddot{\theta}$	ms^2, Js^2
spring	$F = k(x_2 - x_1)$	$T = k_t(\theta_2 - \theta_1)$	k, k_t
damper	$F = c(\dot{x}_2 - \dot{x}_1)$	$T = c_t(\dot{\theta}_2 - \dot{\theta}_1)$	cs, $c_t s$
units	k N/m, c N s/m	k_t N m/rad, c_t N m s/rad	

A spring feels only the difference

It is $k(x_2 - x_1)$, never kx_2. When both ends move, both coordinates appear — with *opposite* signs on the two bodies it connects.

The Newtonian recipe

Handbook § 3.3.1

1. **Count the degrees of freedom**; give each one a coordinate.
2. **Choose a positive direction** for each coordinate — once, and never change it.
3. **Draw one FBD per body.** Every element touching the body is a force on it.
4. **Apply** $\sum F = m\ddot{x}$ (or $\sum M_O = I_O\ddot{\theta}$) to each FBD.
5. **Collect** into $M\ddot{\mathrm{q}} + C\dot{\mathrm{q}} + K\mathrm{q} = \mathrm{f}$ and check the symmetry.
6. **Laplace-transform** with zero ICs and solve for $G(s)$.

Step 2 is where most wrong models are born

A sign error does not look like an error. It looks like a perfectly reasonable equation for a different machine.

Two masses on one mount

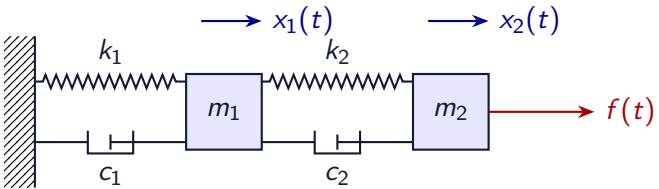
An instrument (m_2) sits on a flexible bracket bolted to a chassis (m_1), which rests on its own mount. A force $f(t)$ acts on the instrument. Handbook § 3.3.2

- How does the instrument displacement x_2 respond to f?
- What does the chassis do meanwhile?
- How many poles will the answer have?

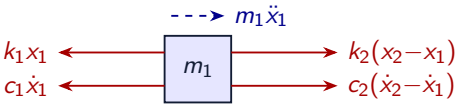
Count first

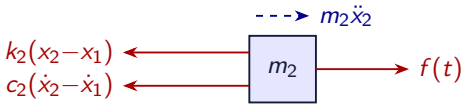
Two masses, two springs: four energy stores. Whatever the algebra does, the denominator of $X_2(s)/F(s)$ must be of degree 4.

Two masses on one mount: the schematic



Two masses, two free-body diagrams





Newton on each mass

Rightward positive for both; $f(t)$ acts on m_2. Handbook § 3.3.2–3.3.3

Mass m_1. Ground elements pull it back; the shared elements pull it towards m_2:

$$m_1\ddot{x}_1 = -k_1x_1 - c_1\dot{x}_1 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1)$$

Why $+k_2(x_2 - x_1)$? If $x_2 > x_1$ the spring is stretched and pulls m_1 to the right — the positive direction.

Mass m_2. The same spring and damper, the other end:

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) + f(t)$$

Equal and opposite on the two bodies — Newton's third law in one line. If your two shared terms have the same sign, stop.

Collect into matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

Handbook § 3.3.3

The free check

M, C and K must be *symmetric*. The off-diagonal $-k_2$ appears twice because one spring touches two bodies. An asymmetric K means a sign was lost on an FBD — go back before you transform anything.

- Diagonal: everything attached to that body.
- Off-diagonal: minus whatever connects the pair.

Laplace turns the matrices into algebra

Zero ICs: each row becomes a polynomial in s.

$$\begin{bmatrix} m_1 s^2 + (c_1 + c_2)s + k_1 + k_2 & -(c_2 s + k_2) \\ -(c_2 s + k_2) & m_2 s^2 + c_2 s + k_2 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ F(s) \end{bmatrix}$$

Cramer's rule for the second unknown:

$$\boxed{\frac{X_2(s)}{F(s)} = \frac{m_1 s^2 + (c_1 + c_2)s + k_1 + k_2}{\Delta(s)}} \qquad \Delta(s) = \det[\,\cdot\,]$$

$\Delta(s)$ is the product of the two diagonal quadratics minus $(c_2 s + k_2)^2$: a *quartic*. Four poles, as the count promised.

A cart driven through springs

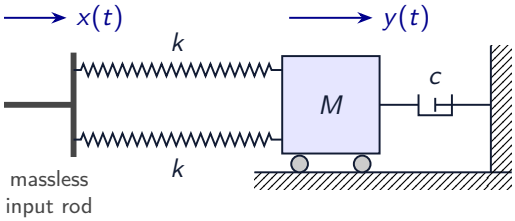
A test rig: an actuator rod moves by $x(t)$. A carriage of mass M follows it through two springs, and a damper ties the carriage to the frame. Handbook § 3.3.6

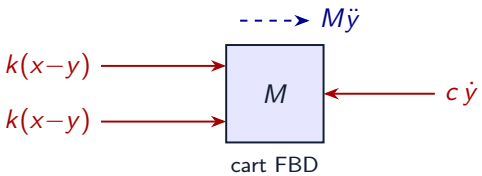
- Does the carriage end up where the rod went?
- How quickly, and does it overshoot on the way?

The input is a displacement, not a force

Many real plants are driven this way — a cam, a rod, a road surface. The input enters *through* the elements, so it appears on the right-hand side multiplied by a stiffness.

Input rod, two springs, one cart





Worked example — input-rod cart (1 of 2)

Reasoning first; no algebra on this slide. Handbook § 3.3.6

Step 1 — the rod. It has no mass, so the net force on it must be zero at every instant. It simply delivers $x(t)$ to the left end of *both* springs.

Step 2 — the springs. Each has deflection $(x - y)$. With the rod ahead ($x > y$) both are compressed and *push* the cart forward, $+k(x - y)$ each. They share both end points: *parallel*, $k_{\text{eq}} = 2k$.

Step 3 — the damper. Its other end is on the frame, not the rod. So its force is $c\dot{y}$, not $c(\dot{x} - \dot{y})$.

Read the picture, not the habit

Step 3 is where a copied FBD from a different rig loses the marks.

Worked example — input-rod cart (2 of 2)

Step 4 — Newton on the cart.

$$M\ddot{y} = 2k(x - y) - c\dot{y} \quad \Longrightarrow \quad M\ddot{y} + c\dot{y} + 2ky = 2k\,x(t)$$

Step 5 — Laplace, zero ICs.

$$\frac{Y(s)}{X(s)} = \frac{2k}{Ms^2 + cs + 2k}, \qquad \omega_n = \sqrt{\frac{2k}{M}}, \qquad \zeta = \frac{c}{2\sqrt{2kM}}$$

Step 6 — put numbers in. $M = 2$ kg, $k = 100$ N/m, $c = 8$ N s/m:

$$2k = 200 \text{ N/m}, \qquad \omega_n = 10 \text{ rad/s}, \qquad \zeta = 0.20$$

$G(0) = 2k/2k = 1$: a 10 mm rod step leaves the cart 10 mm along — after an underdamped overshoot, since $\zeta < 1$.

Rotation: same law, moments instead of forces

$$\sum M_O = I_O\,\ddot{\theta}, \qquad I_O = I_G + m\,d^2 \qquad \text{(parallel axis)}$$

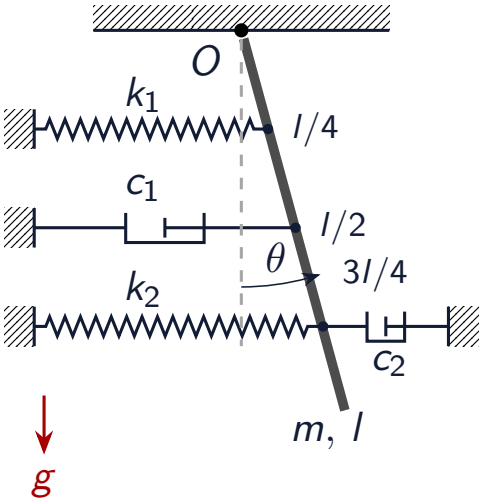
Handbook § 3.2.2, 3.3.7

- For small θ, a point at distance r from the pivot moves sideways by $r\theta$, at speed $r\dot{\theta}$.
- Take moments about the *pivot*. The pivot reaction has zero arm there, so it never enters the equation.
- Gravity is a force like any other: weight mg at the centre of mass, with its own arm.

Engineering question

A bar hangs from a pin, held by springs and dampers at different heights. Which element dominates the motion — the stiff one, or the far one?

A bar held by springs and dampers



Why every moment arm is squared

A spring k attached at distance r from the pivot: Handbook § 3.3.7

Stretch. the attachment point moves $r\theta$
Force. $k\,(r\theta)$, opposing the motion
Moment about O. $k\,(r\theta) \times r = k\,r^2\,\theta$

The same three lines for a damper give $c\,r^2\,\dot{\theta}$.

$$\boxed{k_\theta = k\,r^2, \qquad c_\theta = c\,r^2}$$

Twice as far, four times the effect

One r from the displacement, one from the moment arm. Forgetting either halves the exponent and the answer.

Worked example — pivoted bar

Handbook § 3.3.7

Slender bar m, length l, pinned at the top. k_1 at $l/4$, c_1 at $l/2$, k_2 and c_2 at $3l/4$.

Step 1 — inertia. $I_O = \frac{1}{12} m l^2 + m \left(\frac{l}{2}\right)^2 = \frac{1}{3} m l^2$

Step 2 — square the arms. $k_\theta = \dfrac{k_1 l^2}{16} + \dfrac{9 k_2 l^2}{16}, \qquad c_\theta = \dfrac{c_1 l^2}{4} + \dfrac{9 c_2 l^2}{16}$

Step 3 — gravity. The bar *hangs*: its weight at $l/2$ restores, adding $mg\,l/2$ to k_θ.

Step 4 — assemble, multiply by $16/l^2$.

$$\tfrac{16}{3} m \ddot{\theta} + (4c_1 + 9c_2)\,\dot{\theta} + \left(k_1 + 9k_2 + \tfrac{8mg}{l}\right)\theta = 0$$

Add a torque $T(t)$ at the pin and $\theta(s)/T(s) = 1/(I_O s^2 + c_\theta s + k_\theta)$ — the cart's form again.

The same equation, wired in copper

A sensor filter or a converter output stage: a resistor, an inductor and a capacitor in series, output taken across the capacitor. Handbook § 3.5

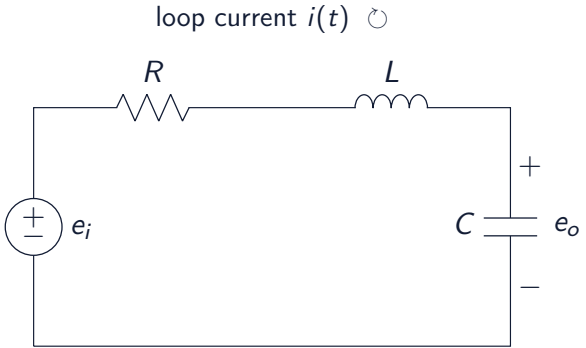
- When the supply switches on, does the output ring?
- How fast does it settle?

Count first: an inductor and a capacitor — two energy stores, so a second-order $G(s)$, exactly like a mass on a spring.

The law changes; the chain does not

Element relations, then Kirchhoff instead of Newton, then Laplace with zero initial conditions.

Series RLC, output across the capacitor



KVL, then impedances

KVL around the loop. Handbook § 3.5.4

$$e_i = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt, \qquad e_o = \frac{1}{C}\int i\,dt$$

Laplace. Elements become impedances R, Ls, $1/(Cs)$, which add in series:

$$E_i(s) = \Big(R + Ls + \frac{1}{Cs}\Big) I(s), \qquad E_o(s) = \frac{I(s)}{Cs}$$

Divide — it is a voltage divider:

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{1/(Cs)}{R + Ls + 1/(Cs)} = \frac{1}{LCs^2 + RCs + 1}}$$

$\omega_n = 1/\sqrt{LC}, \quad \zeta = \frac{R}{2}\sqrt{C/L}.$

Mechanical and electrical: one dictionary

Handbook § 3.5.8

	Mechanical	Series RLC
equation	$m\ddot{x} + c\dot{x} + kx = f$	$L\ddot{q} + R\dot{q} + q/C = e$
inertia	m	L
dissipation	c	R
restoring	k	$1/C$
coordinate, rate	$x, \quad \dot{x}$	$q, \quad i = \dot{q}$

The order rule, again

The two-loop circuit of § 3.5.6 has L, C_1 and C_2: three energy stores, so a *cubic* denominator. If your degree disagrees with your count, the algebra is wrong — not the count.

The plant the rest of the course controls

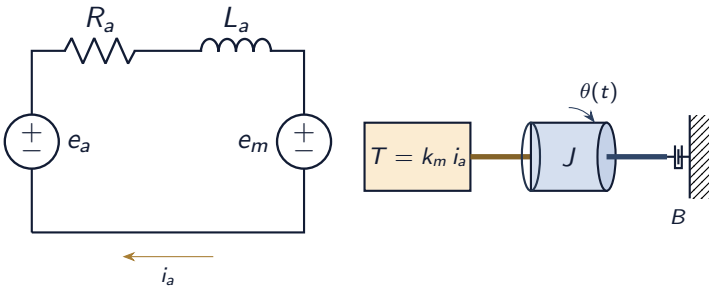
A DC motor turns volts into shaft angle: the solar tracker, a conveyor, a robot joint, a drone. Handbook § 3.7.1

- Given the armature voltage $e_a(t)$, what is the angle $\theta(t)$?
- Which constants on the datasheet actually matter?

Why it earns a whole week

PID tuning, root locus and Bode design in Chapters 4–5 are all done on this plant. It is also the first model that crosses two domains: a circuit and a shaft, coupled by magnetism.

The armature-controlled DC motor



Two sides, one shaft

Electrical side — KVL around the armature. Handbook § 3.7.2

$$e_a = R_a\, i_a + L_a \frac{di_a}{dt} + e_m$$

Mechanical side — Newton for rotation.

$$T = J\,\ddot{\theta} + B\,\dot{\theta}$$

Two equations, four unknowns: i_a, e_m, T, θ.

The magnetism supplies the missing two

Current in a field makes torque. A winding turning in a field makes voltage. Those two coupling equations are the whole of the electromechanics.

Two couplings, one constant

Handbook § 3.7.2, 3.7.9

back-EMF: $e_m = k_b\,\dot{\theta}$ [V s/rad] torque: $T = k_m\,i_a$ [N m/A]

Power balance in the air gap. Mechanical power out equals electrical power converted:

$$T\,\dot{\theta} = e_m\,i_a \implies (k_m i_a)\,\dot{\theta} = (k_b\dot{\theta})\,i_a \implies \boxed{k_m = k_b}$$

Equal only in SI units

A datasheet K_V in RPM/V is not $1/k_b$. Convert: $k_m = k_b = \dfrac{60}{2\pi K_V}$. Skipping the conversion is an error of 9.55 times.

Laplace, eliminate, divide

Step 1 — transform, zero ICs. Handbook § 3.7.3

$$(R_a + L_a s)\, I_a + k_b s\, \theta = E_a, \qquad k_m\, I_a = (J s^2 + B s)\, \theta$$

Step 2 — eliminate I_a using the second equation:

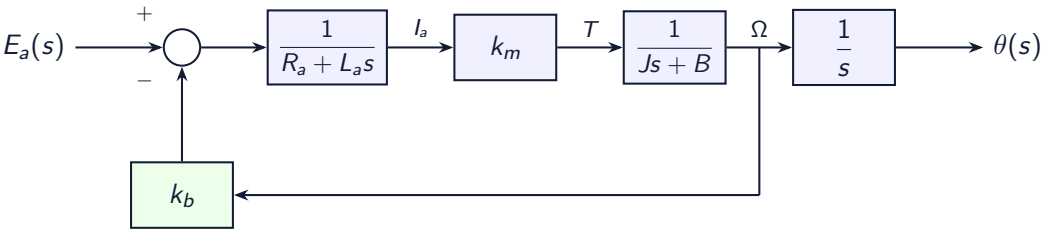
$$E_a = (R_a + L_a s)\frac{(J s^2 + B s)\, \theta}{k_m} + k_b s\, \theta$$

Step 3 — divide.

$$\boxed{\frac{\theta(s)}{E_a(s)} = \frac{k_m}{s\big[(R_a + L_a s)(J s + B) + k_m k_b\big]}} \qquad \frac{\Omega(s)}{E_a(s)} = \frac{k_m}{(R_a + L_a s)(J s + B) + k_m k_b}$$

Third order in angle, second order in speed. The extra $1/s$ is the shaft angle integrating its own speed.

Back-EMF is a feedback loop inside the motor



Dropping the inductance — and when you may

If the armature is electrically fast, set $L_a = 0$: Handbook § 3.7.5

$$\frac{\theta(s)}{E_a(s)} \approx \frac{K}{s(\tau_m s + 1)}, \qquad K = \frac{k_m}{R_a B + k_m k_b}, \qquad \tau_m = \frac{R_a J}{R_a B + k_m k_b}$$

The condition, stated as a number

Compare $\tau_e = L_a/R_a$ with τ_m. Drop L_a only if τ_e is *much* smaller. Many textbook motors qualify.

Check it, do not assume it

The next worked example is a motor that does *not* qualify. See what the simplified model would have told you.

Worked example — open-loop DC motor (1 of 3)

The Case Study 4 motor: $R_a = 1\ \Omega$, $L_a = 0.5$ H, $k_m = k_b = 0.01$ (SI), $J = 0.01\ \text{kg m}^2$, $B = 0.1$ N m s/rad. A 5 V step. Find the speed transfer function, its poles and the steady speed. Handbook § 3.7, C.2

Step 1 — the two time scales, before any algebra.

$$\tau_e = \frac{L_a}{R_a} = 0.5 \text{ s}, \qquad \frac{J}{B} = 0.1 \text{ s}$$

Step 2 — the denominator.

$$(0.5s + 1)(0.01s + 0.1) + 0.01 \times 0.01 = 0.005s^2 + 0.06s + 0.1001$$

Step 3 — make it monic (divide by 0.005):

$$\boxed{\frac{\Omega(s)}{E_a(s)} = \frac{2}{s^2 + 12s + 20.02}}$$

Step 4 — poles. $s^2 + 12s + 20.02 = 0$ gives $s = -2.00$ and $s = -10.0$. Both real: the speed rises without overshoot.

Step 5 — steady speed, two independent ways.

Final-value theorem

$$\omega_{ss} = 5 \times \frac{2}{20.02} = 0.4995 \text{ rad/s}$$

Torque balance ($\dot{\omega} = 0$, $\dot{i}_a = 0$)

$$\omega_{ss} = \frac{k_m V}{R_a B + k_m k_b} = 0.4995 \text{ rad/s}$$

Step 6 — where the volts went. $i_{ss} = B\omega_{ss}/k_m = 4.995$ A. The back-EMF is only $k_b\omega_{ss} = 5.0$ mV: almost the whole 5 V is dropped across R_a.

Two routes, one number

When the transfer function and a physical balance agree, you have checked the algebra without redoing it.

Step 7 — position. $\theta(s)/E_a(s)$ has a pole at $s = 0$. Speed settles; angle does not. Integrating the three motor ODEs numerically gives $\theta = 4.70$ rad at 10 s, still climbing at 0.4995 rad/s.

No constant voltage holds this shaft at an angle. That is the argument for feedback, which Chapter 4 makes.

Could we have dropped L_a?

The simplified model gives $\tau_m = 0.0999$ s: one pole at -10.01. The true slow pole is at -2.00 — *five times* slower, set by $\tau_e = 0.5$ s. Here τ_e is the *longer* time constant, so the simplified model predicts a response five times too fast.

A sister motor — a 12 V screwdriver motor (1 of 2)

Same equations, a different motor: a cordless screwdriver. $R_a = 1.2\ \Omega$, $L_a = 2$ mH, $k_m = k_b = 0.03$ (SI), $J = 4 \times 10^{-5}$ kg m^2, $B = 2 \times 10^{-6}$ N m s/rad. A 12 V step.

Step 1 — the two time scales. $\tau_e = L_a/R_a = 1.67$ ms; $\quad \tau_m = \dfrac{R_a J}{R_a B + k_m k_b} = 53.2$ ms.

Ratio 0.031: here L_a *may* be dropped.

Step 2 — the instant of switch-on. $\omega = 0$, so no back-EMF: the current heads for the stall value $V/R_a = 10.0$ A. L_a holds the peak to 9.19 A at 6.1 ms.

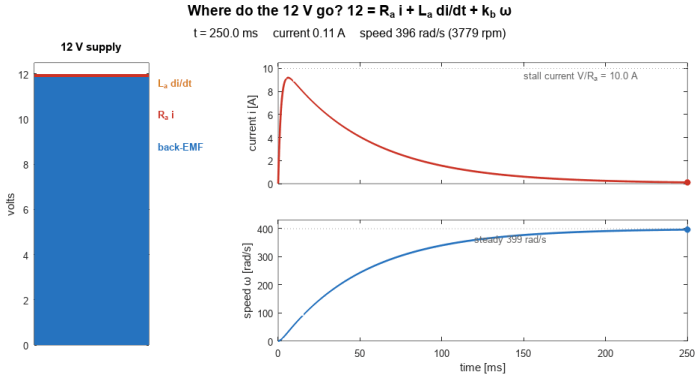
Step 3 — steady running. $\omega_{ss} = \dfrac{k_m V}{R_a B + k_m k_b} = 399$ rad/s (3810 rpm),

$i_{ss} = B\omega_{ss}/k_m = 26.6$ mA, $\quad$ back-EMF $k_b \omega_{ss} = 11.97$ V.

The opposite of Case Study 4

There the back-EMF took 5.0 mV of the 5 V. Here it takes 11.97 V of the 12 V. Check the numbers; never carry a habit from one motor to the next.

Watch the back-EMF take over (2 of 2)



Left: where the 12 V go at each instant. At switch-on it all sits across L_a, then moves to R_a within milliseconds; the back-EMF passes 6 V at 37.6 ms and the speed is within 2 % of 399 rad/s by 203 ms. Poles -19.4 and -581 s^{-1}; with $L_a = 0$, -18.8. A jammed bit keeps $\omega = 0$ and the current at 10 A — that is how a stalled motor burns. GIF: `ch3-anim/MEC524-ch3-motor.gif`.

Gear pair: two constraints, one square law

Handbook § 3.2.5

A small fast motor drives a big slow load. What inertia does the motor feel?

Kinematics — the pitch lines share one speed: $N_1\theta_1 = N_2\theta_2$
Torque — one tooth force acts on both gears: $T_1/N_1 = T_2/N_2$

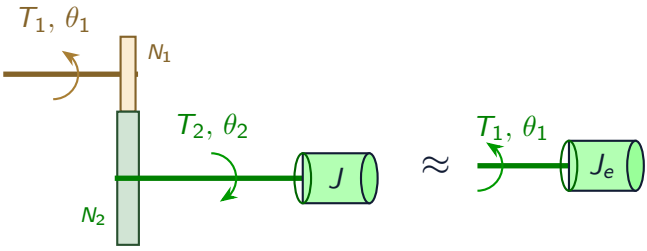
Load on shaft 2. $T_2 = J\ddot{\theta}_2$. Substitute both constraints:

$$\frac{N_2}{N_1}\,T_1 = J\,\frac{N_1}{N_2}\,\ddot{\theta}_1 \quad\Longrightarrow\quad T_1 = J\Big(\frac{N_1}{N_2}\Big)^2\ddot{\theta}_1$$

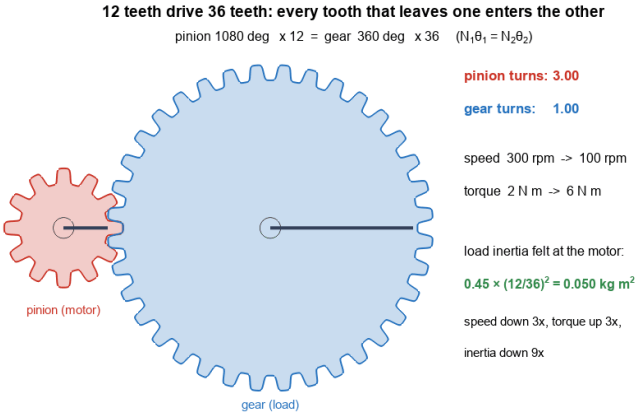
$$\boxed{J_e = J\Big(\frac{N_1}{N_2}\Big)^2}$$

and the same law reflects damping and stiffness.

Reflecting a load through one gear pair



Watch the teeth keep count



A 12-tooth pinion drives a 36-tooth gear: three pinion turns per gear turn, $N_1\theta_1 = N_2\theta_2$ in every frame. 300 rpm in, 100 rpm out; 2 N m in, 6 N m out — the power is unchanged. A 0.45 kg m^2 load feels like $0.45 \times (12/36)^2 = 0.050$ kg m^2 at the motor: nine times lighter. GIF: `ch3-anim/MEC524-ch3-gear.gif`.

The same answer from energy

Kinetic energy is a scalar: it cannot depend on which shaft you stand on. Handbook § 3.2.5

$$\tfrac{1}{2} J \dot{\theta}_2^2 = \tfrac{1}{2} J_e \dot{\theta}_1^2, \qquad \dot{\theta}_2 = \frac{N_1}{N_2}\dot{\theta}_1 \quad \Longrightarrow \quad J_e = J\left(\frac{N_1}{N_2}\right)^2$$

Power gives the torque ratio the same way: $T_1\dot{\theta}_1 = T_2\dot{\theta}_2$, so torque scales as the *inverse* of speed.

Use it as a direction check

The *slow* shaft carries the *big* torque. Reflecting a load from a slow shaft to a fast one must make it look *lighter*. If your number got bigger, the ratio is upside down.

Worked example — reflected inertia

Single pair (§ 3.2.5). $J = 110$ kg m^2 on the gear, $N_1 = 20$ on the input, $N_2 = 100$ on the output.

$$\left(\frac{N_1}{N_2}\right)^2 = 0.04, \qquad J_e = 110 \times 0.04 = 4.4 \text{ kg m}^2$$

A 5:1 reduction makes the load look 25 times lighter.

Two stages (§ 3.7.8). $N_1 = 10$, $N_2 = 35$, $N_3 = 47$, $N_4 = 7$. Ratios *multiply*, stage by stage:

$$\frac{\theta_L}{\theta_m} = \frac{N_3}{N_4} \cdot \frac{N_1}{N_2} = \frac{47}{7} \cdot \frac{10}{35} = 1.918$$

The load shaft turns *faster* than the motor — a step-up. So:

$$J_e = 110 \times 1.918^2 = 404.8 \text{ kg m}^2 \quad \text{— heavier, not lighter.}$$

Square the overall ratio of load speed to motor speed. Which shaft is faster decides lighter or heavier.

Put the gearbox into the motor model

Motor rotor J_a, D_a; load J_L, D_L behind one pair $N_1{:}N_2$. Handbook § 3.7.7

$$J_{\text{eq}} = J_a + J_L\left(\frac{N_1}{N_2}\right)^2, \qquad D_{\text{eq}} = D_a + D_L\left(\frac{N_1}{N_2}\right)^2$$

$$\frac{\theta_m(s)}{E_a(s)} = \frac{k_m}{s\big[(R_a + L_a s)(J_{\text{eq}} s + D_{\text{eq}}) + k_m k_b\big]}, \qquad \theta_L = \frac{N_1}{N_2}\,\theta_m$$

The ratio appears twice, differently

Squared inside J_{eq} and D_{eq}; to the *first* power in the output θ_L. Both are needed for $\theta_L(s)/E_a(s)$.

A drone motor has no viscous load

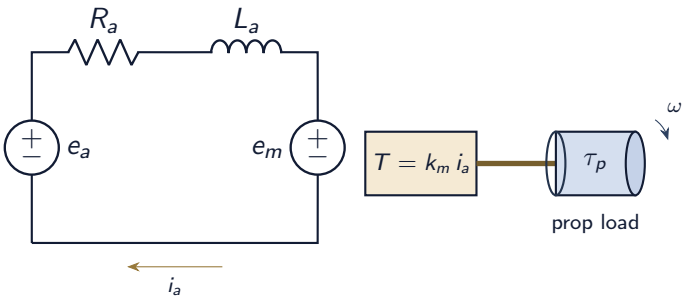
A brushless (BLDC) motor, switched by an electronic speed controller at duty cycle D, turns a propeller. Handbook § 3.7.9

- After ideal commutation the armature loop is the brushed motor's loop, with $e_a = D\,V_{\mathrm{bat}}$.
- But a propeller does not demand $B\omega$. It demands $\tau_p = K_Q\,\omega^2$.

Engineering question

At hover throttle, how fast does the rotor turn, and how much armature current does it draw? Speed sets the vibration frequencies; current sets the wiring and, with the duty cycle, the flight time.

The BLDC loop after ideal commutation



Worked example — BLDC hover speed (1 of 2)

Reasoning first. Handbook § 3.7.9

Step 1 — steady state. Throttle held, so $di_a/dt = 0$ and $\dot{\omega} = 0$. KVL becomes algebraic:

$$i_a = \frac{e_a - k_b\,\omega}{R_a}$$

Step 2 — swap the load. Motor torque now balances the propeller, not a bearing: $k_m\, i_a = K_Q\, \omega^2$

Step 3 — combine. With ω in RPM, $k_b\omega = \omega/K_V$ volts:

$$K_Q\,\omega^2 + \frac{k_m}{R_a K_V}\,\omega - \frac{k_m\, e_a}{R_a} = 0$$

A straight motor line meets a parabolic load curve once. Take the positive root.

Worked example — BLDC hover speed (2 of 2)

$V_{\mathrm{bat}} = 14.8$ V, $K_V = 980$ RPM/V, $R_a = 0.133\ \Omega$, $K_Q = 4.37 \times 10^{-9}$ N m/RPM^2.

Step 4 — the constant. $k_m = 60/(2\pi \times 980) = 9.74 \times 10^{-3}$ N m/A.
Step 5 — hover, $D = 0.50$**.** $e_a = 7.40$ V; the positive root is $\omega = 5490$ RPM.

Step 6 — check it physically.

- back-EMF $5490/980 = 5.60$ V, leaving 1.80 V across R_a;
- $I_a = 1.80/0.133 = 13.5$ A, torque $k_m I_a = 0.132$ N m $= K_Q \omega^2$.

Step 7 — transition, $D = 0.60$**.** 6347 RPM at 18.1 A: 16% more speed costs about a third more current.

A tank is a capacitor that holds water

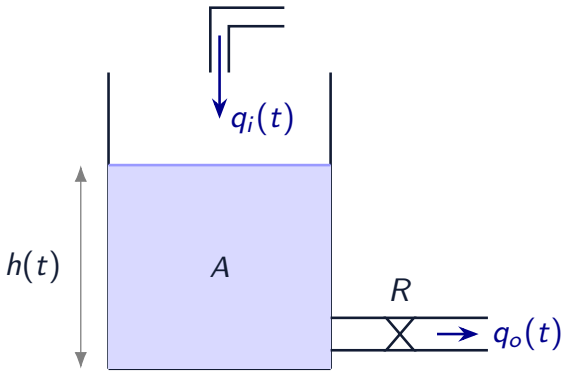
A header tank feeds a process line through a valve. The inflow steps up. Handbook § 3.6

- How far does the level rise?
- How long until it settles?

Same chain, third law

Elements: a tank *stores* (capacitance), a valve *dissipates* (resistance). Law: conservation of mass. One store, so first order.

One tank, one valve



Continuity, then the valve

Law — storage rate = inflow − outflow. Tank area A, so $C = A$: Handbook § 3.6.3

$$C\,\frac{dh}{dt} = q_i - q_o$$

Element — the valve, linearised about the operating point: $q_o = h/R$

$$RC\,\frac{dh}{dt} + h = R\,q_i \quad\Longrightarrow\quad \boxed{\frac{H(s)}{Q_i(s)} = \frac{R}{RCs+1}}, \qquad \tau = RC$$

Every variable is a deviation

A real orifice passes $q \propto \sqrt{h}$. R is the *slope* at the operating point, so h and q here are small changes from it.

Worked example — how long does the level take?

$A = 2\ \text{m}^2$, $R = 50\ \text{s/m}^2$. The inflow steps up by $0.01\ \text{m}^3/\text{s}$. Handbook § 3.6.3

Step 1 — time constant. $\tau = RA = 50 \times 2 = 100\ \text{s}$

Step 2 — final rise. $\Delta h = R\,\Delta q = 50 \times 0.01 = 0.5\ \text{m}$

Step 3 — the clock.

- after $1\tau = 100$ s: $1 - e^{-1} = 63.2\%$, i.e. 0.316 m;
- after $4\tau = 400$ s: $1 - e^{-4} = 98.2\%$ — settled.

Units are the check

s/m^2 times m^3/s is metres. s/m^2 times m^2 is seconds. If either comes out otherwise, R or C was taken in the wrong form.

Two tanks do not simply multiply

Tank 1 drains through R_1 into tank 2, which drains through R_2. Handbook § 3.6.4

$$C_1\dot{h}_1 = q_i - q_1, \quad q_1 = \frac{h_1 - h_2}{R_1}; \qquad C_2\dot{h}_2 = q_1 - q_o, \quad q_o = \frac{h_2}{R_2}$$

Eliminate q_1, h_1; Laplace with zero ICs:

$$\frac{Q_o(s)}{Q_i(s)} = \frac{1}{R_1C_1R_2C_2\,s^2 + (R_1C_1 + R_2C_2 + R_2C_1)\,s + 1}$$

The extra term is loading

If tank 2 did not push back, the answer would be $1/[(R_1C_1s+1)(R_2C_2s+1)]$, with no R_2C_1. Head h_2 back-pressures valve R_1, so the tanks interact. Never multiply two first-order blocks without asking whether the second loads the first.

When Newton hurts

A cylinder rolls without slipping; a spring ties its axle to a wall. Handbook § 3.4.7

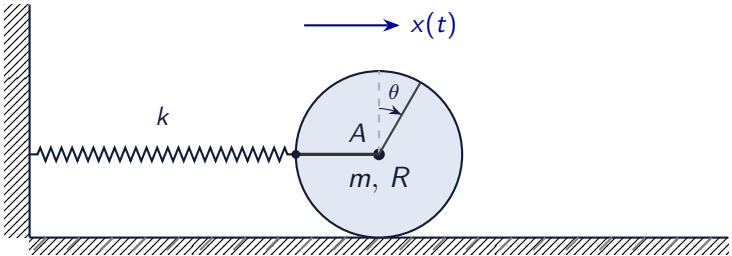
Newton needs $\sum F = m\ddot{x}$ *and* $\sum M_A = J\ddot{\theta}$ — and both contain the friction force at the contact, which nobody asked for. Two equations, three unknowns, plus $x = R\theta$.

Energy does not. Rolling friction does no work, so it never appears. Write T and V in one coordinate and differentiate.

Where the energy method pays

Angles, rolling and linkage constraints, many degrees of freedom: anywhere a reaction force would have to be computed and then eliminated.

A cylinder that rolls, not slides



rough surface: $x = R\theta$

The energy recipe

Kinetic T, potential V, Rayleigh dissipation $\mathcal{D}$, generalised force Q_i — one equation per coordinate: Handbook § 3.4.2–3.4.4

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} + \frac{\partial \mathcal{D}}{\partial \dot{q}_i} = Q_i, \qquad L = T - V$$

Sanity check on a mass–spring–damper. $T = \frac{1}{2}m\dot{x}^2$, $V = \frac{1}{2}kx^2$, $\mathcal{D} = \frac{1}{2}c\dot{x}^2$, $Q = f$:

$$\frac{d}{dt}(m\dot{x}) - (-kx) + c\dot{x} = f \quad \Longrightarrow \quad m\ddot{x} + c\dot{x} + kx = f \ \checkmark$$

Gravity and springs live in V; dampers in $\mathcal{D}$; applied forces and motor torques in Q_i. Nothing else.

Worked example — rolling cylinder

Handbook § 3.4.7

Homogeneous cylinder: mass m, radius R, $J = \frac{1}{2}mR^2$; spring k on the axle.

Step 1 — constraint. No slip: $x = R\theta$, so $\dot{\theta} = \dot{x}/R$.

Step 2 — energies in x alone. $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{\dot{x}}{R}\right)^2 = \frac{3}{4}m\dot{x}^2, \quad V = \frac{1}{2}kx^2$

Step 3 — conservative, so $T + V$ is constant. Differentiate:

$$\tfrac{3}{2}m\dot{x}\ddot{x} + kx\dot{x} = 0 \quad \Longrightarrow \quad m\ddot{x} + \tfrac{2}{3}kx = 0$$

Step 4 — read off. $\omega_n = \sqrt{\dfrac{2k}{3m}} = 0.816\sqrt{\dfrac{k}{m}}$

Rolling adds $\frac{1}{2}m$ of apparent mass along x. The friction force was never computed.

Rolling without slip is everywhere

Handbook § 3.3, 3.4.7

- a trolley, a bicycle or a car wheel on the road;
- a pipe or a drum rolled across a yard;
- a lift or crane drum winding its rope — the rope paid out is the arc the drum has turned;
- the balls inside a bearing, rolling on their races.

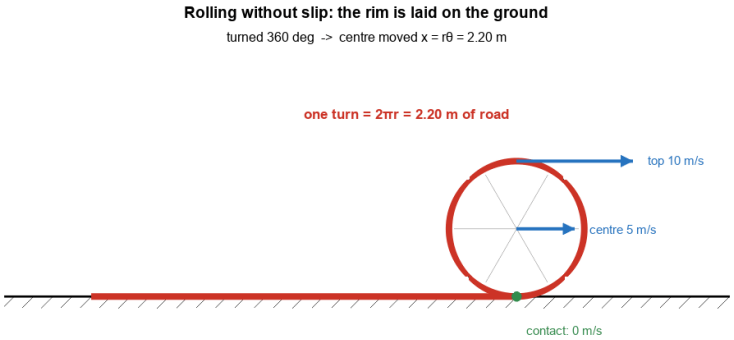
All of them obey one rule: *the length of rim that has touched the ground equals the distance the centre has travelled.*

The rule in one line

$$x = r\theta, \qquad v = r\omega, \qquad a = r\alpha$$

It holds while friction can grip the contact. If the wheel skids, x and θ come apart and you have two coordinates, not one.

Watch a wheel lay its rim on the ground



The red rim is the rubber that has touched the road; the red road is the distance travelled. They are always the same length, so $x = r\theta$. One turn of a 0.35 m bicycle wheel covers $2\pi r = 2.20$ m. GIF: `ch3-anim/MEC524-ch3-roll.gif`.

The contact point is at rest — for an instant

Every point of the wheel = the centre's velocity + its spin about the centre:

$$v_P = v_G + \omega \times r_{GP}$$

	contact	centre	top
speed	$v - r\omega = 0$	v	$v + r\omega = 2v$
bicycle at 18 km/h	0 m/s	5 m/s	10 m/s

Wheel radius 0.35 m, so $\omega = v/r = 14.3$ rad/s.

Two consequences you will use

The contact point does not move, so floor friction does *no work* — that is why the energy method can ignore it. And the top moves twice as far as the centre: a spring tied to the top stretches $2r\theta$ (handbook § 3.3).

Why a rolling body feels heavier: $m + J/r^2$

Kinetic energy = translation + spin. Put $\omega = v/r$:

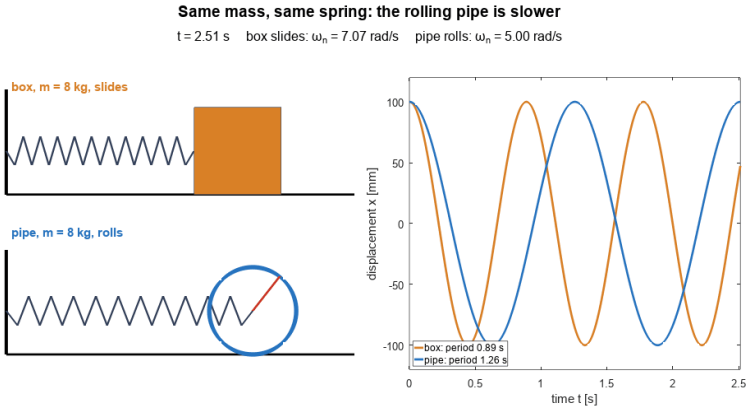
$$T = \tfrac{1}{2}mv^2 + \tfrac{1}{2}J\omega^2 = \tfrac{1}{2}\left(m + \frac{J}{r^2}\right)v^2$$

body	J about the centre	equivalent mass along x
box that slides	—	m
solid roller, disc	$\tfrac{1}{2}mr^2$	$1.5\,m$
thin pipe, hoop	mr^2	$2\,m$

The spring sees the same k; the motion sees more mass

Part of every push goes into spin. Same spring, more apparent mass, so a *lower* natural frequency.

Watch a pipe and a box on the same spring



Same 8 kg, same 400 N/m, both released 100 mm out. The box slides: $\omega_n = 7.07$ rad/s, period 0.89 s. The pipe rolls: half its kinetic energy is spin, so it moves like 16 kg: $\omega_n = 5.00$ rad/s, period 1.26 s. GIF: `ch3-anim/MEC524-ch3-pipe.gif`.

Worked example — a pipe on a spring

A thin-walled pipe, $m = 8$ kg, $r = 0.15$ m, rolls without slip; a spring $k = 400$ N/m is tied to its axle. Find ω_n. Handbook § 3.4.7

Step 1 — constraint. No slip: $x = r\theta$, so $\dot{\theta} = \dot{x}/r$.

Step 2 — inertia of a thin pipe. $J = mr^2 = 8 \times 0.15^2 = 0.18\ \text{kg m}^2$.

Step 3 — energies in x alone. $T = \frac{1}{2}\left(m + J/r^2\right)\dot{x}^2 = \frac{1}{2}(16)\dot{x}^2, \quad V = \frac{1}{2}(400)x^2$

Step 4 — $T + V$ constant; differentiate. $16\ddot{x} + 400x = 0$

Step 5 — read off. $\omega_n = \sqrt{400/16} = 5.00$ rad/s, $\quad f_n = 0.796$ Hz.

Check it two ways

Sliding, it would be $\sqrt{400/8} = 7.07$ rad/s; the ratio is $0.707 = 1/\sqrt{2}$, because the mass doubled. A solid roller of the same mass gives 5.77 rad/s — in between, as $1.5\,m$ says it must.

Rolling: four ways to slip up

- **Mass counted twice.** $m + J/r^2$ uses J about the *centre*. I_O about the contact point already contains mr^2 — do not add m again.
- **Centre or top?** x is the centre. A point on top moves $2r\theta$; a spring tied there sees $2x$.
- **Hoop or disc?** $J = mr^2$ for a thin pipe, $\frac{1}{2}mr^2$ for a solid roller. Read the shape before the formula.
- **No grip, no constraint.** If the body slides, or the friction needed exceeds μN, $x = r\theta$ is gone: keep x and θ as two coordinates.

Traps on the free-body diagram

- **Relative motion.** A spring or damper between two bodies acts with $k(x_2 - x_1)$ — equal and opposite on the two.
- **Which end is grounded.** A damper to the frame gives $c\dot{y}$; a damper to a moving input gives $c(\dot{x} - \dot{y})$.
- **Gravity.** Measure from static equilibrium and weight cancels against spring preload. On a pendulum or hanging bar it does *not* cancel: it adds stiffness (or removes it, if inverted).
- **Moment arms.** Lateral elements on a pivoted body enter as kr^2 and cr^2.
- **Symmetry.** M, C, K asymmetric? A sign was lost. Find it before the Laplace step.

Traps in the motor and the gearbox

- **Square the ratio** for inertia, damping and stiffness — but use it to the first power for the output angle.
- **Upside down.** Reflecting from a slow shaft to a fast one makes the load lighter. A step-up makes it heavier.
- **Back-EMF units.** k_b is V s/rad. From K_V in RPM/V use $60/(2\pi K_V)$. $k_m = k_b$ holds in SI only.
- **Dropping L_a.** Compare $\tau_e = L_a/R_a$ with τ_m first. The Case Study 4 motor fails the test.
- **Angle or speed?** $\theta(s)/E_a(s)$ has the extra $1/s$; $\Omega(s)/E_a(s)$ does not. Read which output is asked for.

Three ways to lose marks here

1. **A sign on a shared spring or damper.**
   The two bodies get equal and opposite forces. An asymmetric stiffness matrix is the evidence.

2. **A reflected inertia not squared, or upside down.**
   $J_e = J(N_1/N_2)^2$. Check the direction: a reduction must make the load lighter at the motor.

3. **A transfer function of the wrong order.**
   Count the energy stores first. The degree of the denominator must match the count.

And one from the motor

k_b taken straight from K_V without converting RPM to rad/s.

What the assessment will ask

Assessment	Week	Weight	What it asks from this chapter
Quiz 2	Wk 7	5% CLO1	Kirchhoff's laws to the transfer function of a circuit; element, motor and gear relations in the MCQ part.
Test 1	Wk 9	30% CLO1+CLO2	Schematic to EOM to transfer function, with motors and gears.
PBL Milestone 1	Wk 10	10% CLO3	A justified lumped model of your plant and its $G(s)$, checked by simulation.

What a full-mark answer looks like

One labelled FBD per body; the law before the numbers; zero ICs stated; the order of $G(s)$ checked against the energy stores; units throughout.

What to take away

1. One chain for every domain: sketch $\rightarrow$ law $\rightarrow$ ODE $\rightarrow$ Laplace (zero ICs) $\rightarrow$ $G(s)$.
2. A shared spring or damper acts equal and opposite on both bodies; symmetric M, C, K confirm it.
3. Pivoted bodies take kr^2 and cr^2; gears reflect as $(N_1/N_2)^2$.
4. The DC motor is KVL + Newton + two couplings, with $k_m = k_b$ in SI. Drop L_a only after comparing L_a/R_a with τ_m.
5. The order of $G(s)$ is the number of independent energy stores.

Handbook § 3.0–3.7. Worked examples: § 3.3 and § 3.5–3.7.

Tutorial: § 3.8 — attempt it before the tutorial class.