Introduction to Control Systems

feedback · Laplace transform · partial fractions · transfer functions · PID

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MEC524 Control Systems · Handbook Chapter 1 · Syllabus topic 1 · Weeks 1–2

Semester 20264

How to use this chapter

Three documents, three different jobs. Use the right one.

These slides	Where every result *comes from*, and what it means. Follow the derivations here.
The handbook	The reference. Look up a transform pair, a table, a worked example. Chapter 1, § 1.0–1.6.
The tutorial	Where you find out whether you actually understood it. § 1.6.

The small print on each slide

Handbook § 1.2 in a corner tells you where the same material sits in the handbook. Read that section *after* the lecture, not instead of it.

Where Chapter 1 is going

By the end of week 2 you should be able to take a plant and answer *four* questions about it.

1. **Is it controlled at all?** Open loop or closed loop — and what the sensor is for. §1.0, 1.4
2. **How do I turn its ODE into algebra?** The Laplace transform. §1.1
3. **How do I get back to time?** Partial fractions. §1.2
4. **What does the plant do on its own, and what does a PID do to it?** §1.3, 1.5

Chapters 2–5 are written in s

Every block diagram, root locus and Bode plot later in MEC524 assumes you can move between t and s without stopping to think.

You already run feedback loops

Before any equation: you have been a controller all your life.

You are…	**you sense**	**you want**	**you change**
in the shower	water on your skin	warm, not hot	the tap
keeping a car in lane	where the lines are	the middle	the steering
on cruise control, uphill	the speedometer	the set speed	the throttle

Same three moves every time

Sense what is happening, **compare** it with what you want, **act** on the difference. Then do it again, and again. That repeated circle is a *feedback loop*. This chapter turns it into mathematics.

The shower: why acting on old news makes you swing

The hot water takes a few seconds to travel from the tap to your head.

1. Too cold. You turn the tap towards hot. Nothing happens yet.
2. Still cold, so you turn it *more*. Still nothing.
3. The first hot water arrives — then all the extra you asked for. Too hot!
4. You swing the tap back hard. A few seconds later: too cold again.

What went wrong

Acting *hard* on a measurement that is *late* makes the output overshoot and swing.

What a good controller does

Smaller corrections, or patience — exactly the trade the P, I and D knobs set at the end of this chapter.

Eight words, one car

Cruise control on a motorway. Every word below appears on every block diagram in MEC524.

Word	In the car	Symbol
plant	the car itself	G
output	the actual speed	y
setpoint (reference)	the speed you set	r
sensor	the speedometer	H
error	set speed $-$ actual speed	$e = r - y$
controller	the cruise-control box	C
actuator	the throttle	u is its command
disturbance	a hill, a headwind	d

Open loop has no sensor and no error: it acts on a guess.

A heater and a thermostat

A plain electric heater

- Switch on, heat comes out.
- The room ends up at whatever temperature the weather allows.
- Nothing is measured, so nothing is corrected.

A thermostat

- Measures the room temperature.
- Compares it with the setpoint.
- Switches the heater to close the gap — whatever the weather does.

The question every control engineer is paid to answer

How do we force a plant's *actual* output to match the *desired* output, as closely and as quickly as possible, while disturbances we cannot predict keep pushing it away?

Why bother: **precision** · **safety** · **efficiency** · **autonomy**

Handbook § 1.0.1–1.0.2

Sense, compare, act: a hobby servo

Command the arm to $r = 45°$. A potentiometer reads the actual angle y; a driver chip computes $e = r - y$ and drives the motor.

Arm reads y	**Error $e = r - y$**	**Controller does**
$44°$	$+1°$	drives towards $45°$
$45°$	0	stops
$46°$ (pushed by hand)	$-1°$	drives back the other way

The controller never needs to know *why* the arm moved

Only the sign and size of e. That is what lets one loop reject a disturbance nobody modelled. A six-axis robot arm is six of these loops.

Handbook § 1.0.4

One architecture, many plants

Plant	Sensor	Actuator
Watt's steam engine	flyball governor	steam valve
Room air	thermostat	heater / HVAC vent
Hard-disk spindle	speed sensor	spindle motor
Car on a motorway	speedometer, radar	throttle and brakes
Quadrotor	gyroscope	rotor thrust

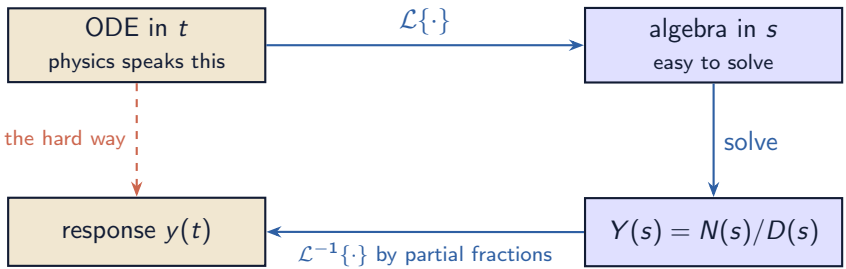
Feedback is not a circuit trick

The governor is built from hinges and weights; the drone from silicon. The block diagram is identical: measure, compare with what you want, act on the difference. It is a property of the *structure*, not of the material.

Handbook § 1.0.4

Why an engineer needs a translator

Physics hands you an ODE in time. Solving it directly — with initial conditions and a messy input — is slow and error-prone.



Transform $\rightarrow$ solve the algebra $\rightarrow$ transform back. Every calculation in MEC524 follows this loop.

Handbook § 1.1, 1.2.1

The definition, and what s means

For a signal $f(t)$ that starts at $t = 0$:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

The variable is complex, $s = \sigma + j\omega$, so the kernel splits in two:

$$e^{-st} = \underbrace{e^{-\sigma t}}_{\text{decay}}\ \underbrace{e^{-j\omega t}}_{\text{oscillation}}$$

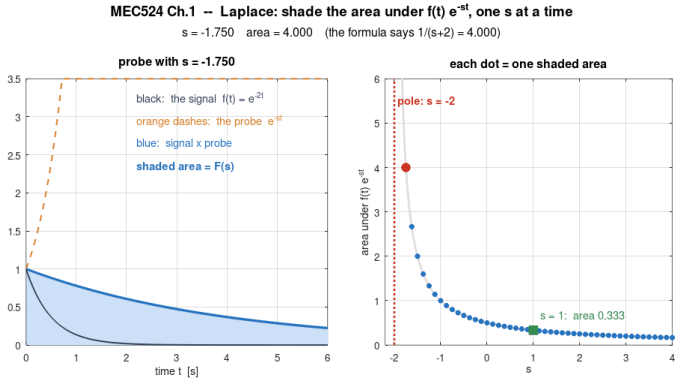
- σ measures exponential decay or growth; ω measures how fast it rings.
- s has units of s^{-1}, so st is dimensionless.

Why this particular integral

It is linear, and it turns d/dt and $\int dt$ — exactly the operations an equation of motion is built from — into multiplication and division.

Handbook § 1.1.1

Watch s at work: one shaded area per value of s



Multiply $f(t) = e^{-2t}$ by the probe e^{-st} and shade what is left: that area *is* $F(s)$. At $s = 1$ it is $0.333 = 1/(1+2)$. Near $s = -2$ the probe stops killing the signal and the area runs away — that is what a **pole** is. GIF: `ch1-anim/MEC524-ch1-probe.gif`.

Three pairs, straight from the integral

Unit step $f = 1$:

$$\int_0^\infty e^{-st}\,dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s}$$

Exponential $f = e^{-at}$: the two exponentials merge,

$$\int_0^\infty e^{-(s+a)t}\,dt = \frac{1}{s+a} \qquad \text{— a pole at } s = -a$$

Ramp $f = t$: integrate by parts ($u = t$, $dv = e^{-st}dt$),

$$\left[-\frac{t\,e^{-st}}{s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt = \frac{1}{s^2} \qquad \text{— a double pole at the origin}$$

The pattern to notice

A signal's rate of decay or growth becomes the *location of a pole*.

Handbook § 1.1.1

The short table you should own by week 2

$f(t)$	$F(s)$	
$\delta(t)$	1	impulse
1	$\dfrac{1}{s}$	step
t^n	$\dfrac{n!}{s^{n+1}}$	polynomial
e^{at}	$\dfrac{1}{s-a}$	exponential

$f(t)$	$F(s)$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$e^{at}\cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
$e^{at}\sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

Right-hand column: cosine has s on top, sine has ω. The bottom two rows are the top two with s replaced by $s-a$ — which the next slides explain.

Handbook § 1.1.2, Table 1.1

Differentiation becomes multiplication by s

Integrate the transform of a derivative by parts:

$$\mathcal{L}\{\dot{x}\} = \int_0^\infty \dot{x}\, e^{-st}\, dt = \Big[x\, e^{-st}\Big]_0^\infty + s\int_0^\infty x\, e^{-st}\, dt = sX(s) - x(0)$$

Apply it twice:

$$\mathcal{L}\{\ddot{x}\} = s^2X(s) - s\,x(0) - \dot{x}(0)$$

With zero initial conditions

$$\frac{d^n}{dt^n} \longmapsto s^n \qquad \text{— an ODE becomes a polynomial in } s.$$

This one line is the reason transfer functions exist (§ 1.3).

Handbook § 1.1.3

Integration, and the two shift theorems

Integration — the mirror image of differentiation:

$$\mathcal{L}\Big\{\int_0^t f(\tau)\,d\tau\Big\} = \frac{F(s)}{s}$$

s-shift — an exponential multiplier just moves s:

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty f(t)\,e^{-(s-a)t}\,dt = F(s-a)$$

t-shift — a pure delay T: $\quad \mathcal{L}\{f(t-T)\,u(t-T)\} = e^{-Ts}F(s)$

A damped sinusoid is a shifted sinusoid

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4} \quad\Longrightarrow\quad \mathcal{L}\{e^{-t}\sin 2t\} = \frac{2}{(s+1)^2+4}$$

Handbook § 1.1.3

Worked — transforms from the table (1 of 2)

Find $\mathcal{L}\{7 + 2e^{-3t}\}$.

Reasoning. A sum: use linearity. Each piece is already a table entry (step, exponential).

$$F(s) = \frac{7}{s} + \frac{2}{s+3}$$

Find $\mathcal{L}\{e^{2t}\cosh t\}$.

Reasoning. The e^{2t} is a *multiplier*, so this is an s-shift with $a = 2$ — not a product of two transforms.

$$\mathcal{L}\{\cosh t\} = \frac{s}{s^2 - 1} \quad \xrightarrow{s \to s-2} \quad F(s) = \frac{s-2}{(s-2)^2 - 1}$$

Handbook § 1.6.1, Problems 1 and 5

Find $\mathcal{L}\{\sin^2 4t\}$.

Reasoning. There is no table entry for a square. Rewrite with an identity first, so that only table entries remain.

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \quad \Longrightarrow \quad \sin^2 4t = \tfrac{1}{2} - \tfrac{1}{2}\cos 8t$$

Arithmetic.

$$F(s) = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s}{s^2+64} = \frac{(s^2+64)-s^2}{2s(s^2+64)} = \frac{32}{s(s^2+64)}$$

The trap

$\mathcal{L}\{f\,g\} \neq F(s)\,G(s)$. A product in time is *not* a product in s. The operation that does become a product is convolution.

Handbook § 1.6.1, Problem 7

Convolution: why $Y(s) = G(s)\,U(s)$

The convolution of two signals:

$$(g * u)(t) = \int_0^t g(\tau)\, u(t-\tau)\, d\tau$$

It is exactly how a linear system builds its output: every past kick of the input, weighted by how the system remembers it. If g is the impulse response,

$$y(t) = (g * u)(t)$$

Convolution theorem

$$\mathcal{L}\{(g * u)(t)\} = G(s)\, U(s)$$

An awkward integral in time becomes a plain product in s.
From Chapter 2 on we write $Y = GU$ without comment — this is why we may.

Handbook § 1.1.4

Getting back to time: sort the poles

Solving the algebra leaves a rational function $Y(s) = N(s)/D(s)$. The table only inverts small pieces, so break Y into pieces. Factor $D(s)$ and sort its roots into three piles:

Pile	Fraction	Time signal
distinct real p	$\dfrac{A}{s-p}$	$A\,e^{pt}$
repeated p (twice)	$\dfrac{A_1}{s-p} + \dfrac{A_2}{(s-p)^2}$	$(A_1 + A_2 t)\,e^{pt}$
complex $\sigma \pm j\omega$	$\dfrac{A(s-\sigma) + B\omega}{(s-\sigma)^2 + \omega^2}$	$e^{\sigma t}(A\cos\omega t + B\sin\omega t)$

Handbook § 1.2.2, Table 1.2

The cover-up rule — where it comes from

Suppose

$$F(s) = \frac{N(s)}{(s-p_1)(s-p_2)} = \frac{A}{s-p_1} + \frac{B}{s-p_2}$$

Multiply both sides by $(s-p_1)$:

$$(s-p_1)\,F(s) = A + B\,\frac{s-p_1}{s-p_2}$$

Now set $s = p_1$. The second term vanishes:

$$A = \Big[(s-p_1)\,F(s)\Big]_{s=p_1}$$

In words

Cover the factor, substitute the pole. It works for every *distinct* pole. For a repeated pole it gives only the highest-power coefficient.

Handbook § 1.2.3

Worked — distinct real poles

Invert $F(s) = \dfrac{5}{s^2 + 3s + 2}$.

Step 1 — factor and sort. $s^2 + 3s + 2 = (s+1)(s+2)$: two distinct real poles, so $F = \dfrac{A}{s+1} + \dfrac{B}{s+2}$.

Step 2 — cover-up.

$$A = \frac{5}{-1+2} = 5, \qquad B = \frac{5}{-2+1} = -5$$

Step 3 — invert each piece.

$$f(t) = 5\,e^{-t} - 5\,e^{-2t}, \quad t \geq 0 \qquad \text{(time constants 1 s and 0.5 s)}$$

Step 4 — one line of check. The denominator is two degrees above the numerator, so the response must start from zero: $f(0) = A + B = 0$.

Handbook § 1.2.3; 1.6.2, Problem 10

Worked — a repeated pole (1 of 2)

Invert $F(s) = \dfrac{2}{(s+1)(s+2)^2}$.

Step 1 — sort. $s = -1$ is distinct; $s = -2$ is double. A double pole needs *two* fractions:

$$F(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

Step 2 — cover-up gives A and C.

$$A = \frac{2}{(-1+2)^2} = 2, \qquad C = \Big[(s+2)^2 F\Big]_{s=-2} = \frac{2}{-2+1} = -2$$

Why cover-up cannot give B. Multiply by $(s+2)^2$ and the B term becomes $B\,(s+2)$ — which vanishes at $s = -2$ and takes B with it. Differentiate first, then substitute.

Handbook § 1.2.4

Step 3 — differentiate, then substitute.

$$B = \frac{d}{ds}\Big[(s+2)^2 F\Big]_{s=-2} = \frac{d}{ds}\Big[\frac{2}{s+1}\Big]_{s=-2} = \Big[\frac{-2}{(s+1)^2}\Big]_{s=-2} = -2$$

Step 4 — invert. The $1/(s+2)^2$ term is the shifted ramp, $t\,e^{-2t}$:

$$f(t) = 2\,e^{-t} - 2\,e^{-2t} - 2\,t\,e^{-2t}$$

Step 5 — check. $f(0) = 2 - 2 - 0 = 0$, as the degree gap demands; and the simple-pole coefficients again sum to zero, $A + B = 0$.

What the $t\,e^{-2t}$ term looks like

It rises before it decays — the fingerprint of a repeated real pole.

Handbook § 1.2.4; 1.6.2, Problem 11

Worked — complex poles: complete the square

Invert $F(s) = \dfrac{3s - 137}{s^2 + 2s + 401}$.

Step 1 — do not factor into complex numbers. The discriminant is negative, so complete the square instead:

$$s^2 + 2s + 401 = (s+1)^2 + 20^2 \qquad (\sigma = -1,\ \omega = 20\ \mathrm{rad/s})$$

Step 2 — write the numerator in the same shift.

$$3s - 137 = 3(s+1) - 140 = 3(s+1) - 7 \times 20$$

Step 3 — read off the shifted cosine and sine.

$$f(t) = e^{-t}\,(3\cos 20t - 7\sin 20t)$$

Check. $f(0) = 3$, the leading numerator coefficient — right for a denominator one degree higher.

Handbook § 1.2.5

Worked — a suspension under road forcing (1 of 3)

A quarter-car suspension, started from rest, is driven by the road:

$$\ddot{x} + 6\dot{x} + 25x = 4\sin 2t$$

The engineering question: what motion does the body settle into, and what does the start-up transient look like?

Step 1 — transform (zero initial conditions):

$$(s^2 + 6s + 25)\,X(s) = \frac{4 \times 2}{s^2 + 4} \quad \Longrightarrow \quad X(s) = \frac{8}{\big((s+3)^2 + 4^2\big)\big(s^2 + 2^2\big)}$$

Step 2 — whose poles are whose?

- $s = -3 \pm j4$ — the suspension's own: they will decay.
- $s = \pm j2$ — the road's: they will not.

Handbook § 1.2.5; 1.6.3, Problem 12

Step 3 — one fraction per quadratic, each with a linear numerator:

$$X(s) = \frac{As + B}{(s+3)^2 + 16} + \frac{Cs + D}{s^2 + 4}$$

Step 4 — clear denominators and match powers of s:

$$(As + B)(s^2 + 4) + (Cs + D)(s^2 + 6s + 25) = 8$$

$$s^3\text{:}\ \ A + C = 0 \qquad s^2\text{:}\ \ B + 6C + D = 0$$
$$s^1\text{:}\ \ 4A + 25C + 6D = 0 \qquad s^0\text{:}\ \ 4B + 25D = 8$$

Step 5 — solve the four equations:

$$A = \frac{16}{195}, \quad B = \frac{40}{195}, \quad C = -\frac{16}{195}, \quad D = \frac{56}{195}$$

Handbook § 1.6.3, Problem 12

Step 6 — invert each quadratic (multiply through by 195):

$$\frac{16s+40}{(s+3)^2+16} = 16\,\frac{s+3}{(s+3)^2+16} - 2\,\frac{4}{(s+3)^2+16} \qquad \text{(subtract } 3\times 16\text{, divide by 4)}$$

$$\frac{-16s+56}{s^2+4} = -16\,\frac{s}{s^2+4} + 28\,\frac{2}{s^2+4} \qquad \text{(divide 56 by 2)}$$

$$x(t) = \frac{1}{195}\Big[\underbrace{e^{-3t}\,(16\cos 4t - 2\sin 4t)}_{\text{transient, dies out}} + \underbrace{(-16\cos 2t + 28\sin 2t)}_{\text{steady state, follows the road}}\Big]$$

Use the initial conditions as a free check

$x(0) = (16-16)/195 = 0$ and $\dot{x}(0) = (-48-8+56)/195 = 0$.
A wrong coefficient anywhere shows up here, in one line.

Handbook § 1.2.5; 1.6.3, Problem 12

From ODE to transfer function

A linear, time-invariant plant with input u and output y:

$$a_n y^{(n)} + \cdots + a_1 \dot{y} + a_0 y = b_m u^{(m)} + \cdots + b_0 u$$

Transform with zero initial conditions — every d/dt becomes s:

$$(a_n s^n + \cdots + a_0)\, Y(s) = (b_m s^m + \cdots + b_0)\, U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + \cdots + b_0}{a_n s^n + \cdots + a_0}, \qquad n \geq m$$

A property of the plant alone

$G(s)$ depends on the physical parameters, not on the input you apply.

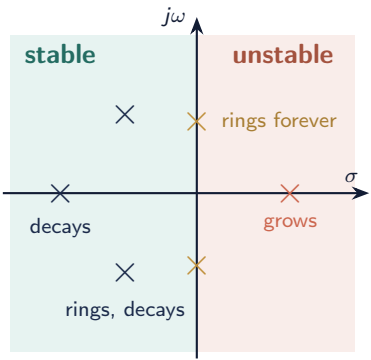
Spring–mass–damper: $m\ddot{y} + c\dot{y} + ky = f \;\Rightarrow\; G(s) = \dfrac{1}{ms^2 + cs + k}$.

Handbook § 1.3.1

Poles decide what the plant does

Expand any response in partial fractions: every pole p of $G(s)$ contributes a term $A\,e^{pt}$.

- $\mathrm{Re}(p) < 0$: the term decays.
- $\mathrm{Re}(p) = 0$: it persists — a constant or a steady ring.
- $\mathrm{Re}(p) > 0$: it grows without bound.

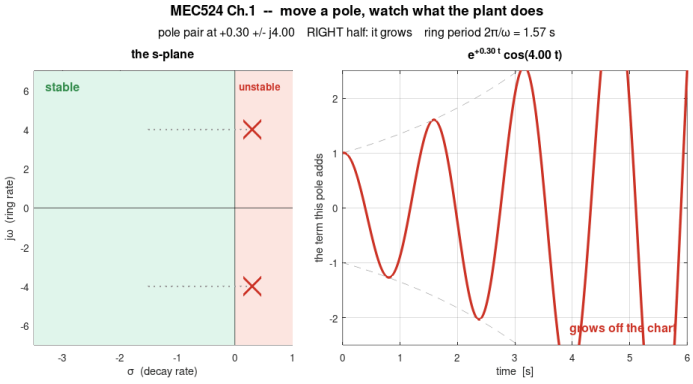


Stability rule

Stable if and only if every pole lies in the open *left* half-plane. Simple poles on the $j\omega$ axis: marginal. A repeated one, or one on the right: unstable.

Handbook § 1.3.2–1.3.3

Watch a pole move



Left to right: one real pole slides from -3 to $+0.5$; then a pair rings faster as ω grows; then a pair at $\pm j4$ crosses the axis. How far *left* sets how fast it dies; how far *up* sets how fast it rings. GIF: `ch1-anim/MEC524-ch1-poles.gif`.

Reading a pole without algebra

A plant has a pole pair at $s = -1 \pm j4$. What will you see on a scope?

Step 1 — how fast it rings (the height, $\omega = 4\,\text{rad/s}$):

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = 1.57\ \text{s per cycle}$$

Step 2 — how fast it dies (the distance left, $\sigma = -1$): the envelope is e^{-t}, so it halves every $\ln 2/1 = 0.69\,\text{s}$.

Step 3 — when it is gone. After $4/|\sigma| = 4\,\text{s}$ the envelope is $e^{-4} = 0.018$, under 2%: about two and a half rings, then quiet.

Same height, other side

At $s = +0.2 \pm j4$ it rings at the same rate but the envelope *doubles* every $\ln 2/0.2 = 3.47\,\text{s}$. Nothing in the numerator can fix that.

Handbook § 1.3.2–1.3.3

Worked — an engine mount as a transfer function (1 of 2)

An engine mount: $m = 1.2\,\text{kg}$, $c = 22\,\text{N s/m}$, $k = 1920\,\text{N/m}$, driven by $F \sin \omega t$ with $F = 80\,\text{N}$ at $\omega = 40\,\text{rad/s}$.

Step 1 — the transfer function (from $m\ddot{x} + c\dot{x} + kx = f$):

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{1.2\,s^2 + 22\,s + 1920}$$

Step 2 — divide by m to reach standard form.

$$G(s) = \frac{1/m}{s^2 + (c/m)\,s + k/m} \quad \text{vs} \quad \frac{\cdots}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Step 3 — match terms.

$$\omega_n = \sqrt{k/m} = \sqrt{1600} = 40\ \text{rad/s}, \qquad 2\zeta\omega_n = c/m = 18.33\ \text{s}^{-1} \ \Rightarrow\ \zeta = 0.229$$

Handbook § 1.6.4, Problem 15

Worked — an engine mount as a transfer function (2 of 2)

Step 4 — locate the poles.

$$s = -\zeta\omega_n \pm j\,\omega_n\sqrt{1-\zeta^2} = -9.17 \pm j\,38.9\ \mathrm{s}^{-1}$$

Step 5 — read the s-plane.

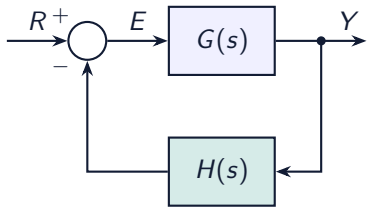
- Both poles in the left half-plane $\Rightarrow$ stable.
- Complex pair, $\zeta < 1$ $\Rightarrow$ underdamped: it rings as it settles.

Now look at the forcing

The engine drives the mount at 40 rad/s — exactly ω_n. A stable, lightly damped plant driven at its natural frequency is why this mount needs an active controller. That design is a Chapter 5 exercise.

Handbook § 1.6.4, Problem 15

Block diagrams and the feedback formula



block = multiply · circle = add with signs · dot = take-off

Write what each primitive says:

$$E = R - HY, \qquad Y = GE$$

Substitute the first into the second:

$$Y = GR - GHY$$

Collect Y:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Perfect sensor: $H = 1$, *unity feedback*.

Handbook § 1.3.4

The toaster cannot see the bread

Open loop

The control action does not depend on the output. One path from input to output; nothing measured, nothing fed back.

- **Toaster.** Set the dial, push the lever. Frozen bread comes up pale; stale bread comes up burnt. The timer cannot tell.
- **Cloth dryer.** Runs for the time you set, however wet the clothes still are.
- **Timed traffic light.** Cycles on schedule, whatever the queue.

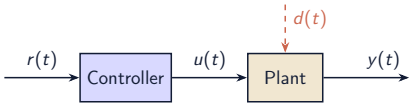
When open loop is the right answer

Small disturbances, a well-known plant that does not drift, and a tight budget. A microwave oven does not need a sensor that costs more than the error it would remove.

Handbook § 1.4.1

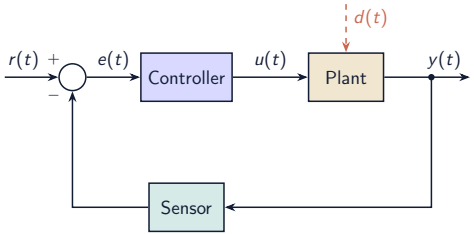
Open loop and closed loop, side by side

(a) OPEN-LOOP



output has no effect on the control action

(b) CLOSED-LOOP (FEEDBACK)



output is sensed and compared to setpoint; error $e = r - y$ drives the controller

A turntable on a draining battery

A DC motor turns a platter, fed from a battery through a variable resistor.

Open loop

- Battery drains, voltage sags.
- Motor slows, platter slows.
- Nothing notices.

Closed loop

- **Sense:** a tachometer reads the speed.
- **Compare:** error = desired − actual.
- **Act:** the error grows, so the controller raises the voltage.

The engineering question

The battery has effectively changed the plant. *How much* does feedback protect the output from a plant that drifts? Next slide: in numbers.

Handbook § 1.4.2

Feedback in numbers: a plant whose gain drifts

Steady state only. The plant's gain K falls from 10 to 8 — a 20% drop. We want $y = 1$.

Open loop. Preset $u = 1/K = 0.1$.

$$K = 10\text{: } y = 1.0 \qquad \longrightarrow \qquad K = 8\text{: } y = 0.8 \qquad \text{(a 20\% error)}$$

Closed loop. Controller gain 10, unity feedback, so $T = \dfrac{10K}{1 + 10K}$ from the feedback formula:

$$K = 10\text{: } y = \frac{100}{101} = 0.990 \qquad \longrightarrow \qquad K = 8\text{: } y = \frac{80}{81} = 0.988$$

A 20% plant drift became a 0.25% output drift

Large loop gain makes $T = GC/(1 + GC) \approx 1$ whatever the plant does: *reduced sensitivity to parameter variations*.

Handbook § 1.4.2

Disturbances and the sensitivity function

Add a disturbance D at the output of a unity-feedback loop:

$$Y = GC\,E + D, \qquad E = R - Y$$

Eliminate E and collect Y:

$$Y = \underbrace{\frac{GC}{1+GC}}_{T(s)}\,R \;+\; \underbrace{\frac{1}{1+GC}}_{S(s)}\,D$$

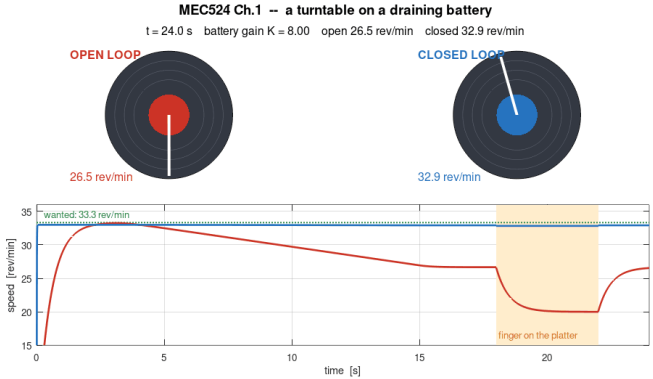
- Open loop, the whole of D reaches the output.
- Closed loop with a loop gain of 100: $S = 1/101$, so about 1% of a steady disturbance survives.

High loop gain is the lever

Large GC makes S small and T close to 1. Chapter 4 makes this precise — and the next slide says why you cannot just turn the gain up.

Handbook § 1.4.3

Watch the battery drain: both numbers come true



Same plant, same gains as the last two slides: K falls from 10 to 8, controller gain 10. Open loop sags from 33.3 to 26.7 rev/min; closed loop from 33.0 to 32.9. A finger on the platter costs the open loop 6.67 rev/min and the closed loop 0.08 — the $S = 1/81$ of the disturbance that survives. GIF: `ch1-anim/MEC524-ch1-turntable.gif`.

Feedback is not free

What feedback buys

- Less sensitivity to plant drift.
- Disturbances actively rejected.
- A faster, better-shaped transient.
- Smaller steady-state error.

What it costs

- New dynamics: overshoot, oscillation, even instability.
- Sensor noise is fed straight into the loop.
- Gain must roll off at high frequency, which caps the bandwidth.
- A sensor, a controller, calibration: money.

Two architectures, two failure modes

An open-loop system can be *wrong*, and is only as stable as its plant, but no loop can push it into oscillation. A closed-loop system is right on average — tuned badly, it rings or goes unstable. The engineer chooses which failure is acceptable.

Handbook § 1.4.4

What should a controller do with the error?

A motor speed loop. At every instant the controller knows $e(t) = r - y$. It can use three things about that error:

Use	Time	Term	Instinct
how big it is *now*	present	$K_p\, e$	push harder when further away
how much has *built up*	past	$K_i \int_0^t e\, d\tau$	do not tolerate a lasting gap
where it is *heading*	future	$K_d \dfrac{de}{dt}$	brake before you overshoot

Add the three and you have the most widely used controller in industry: **PID**.

Handbook § 1.5

The PID law in the s-domain

In time:

$$u(t) = K_p\, e(t) + K_i \int_0^t e(\tau)\, d\tau + K_d \frac{de(t)}{dt}$$

Transform with zero initial conditions, using the two properties from § 1.1 — integration divides by s, differentiation multiplies by s:

$$U(s) = \left(K_p + \frac{K_i}{s} + K_d\, s \right) E(s)$$

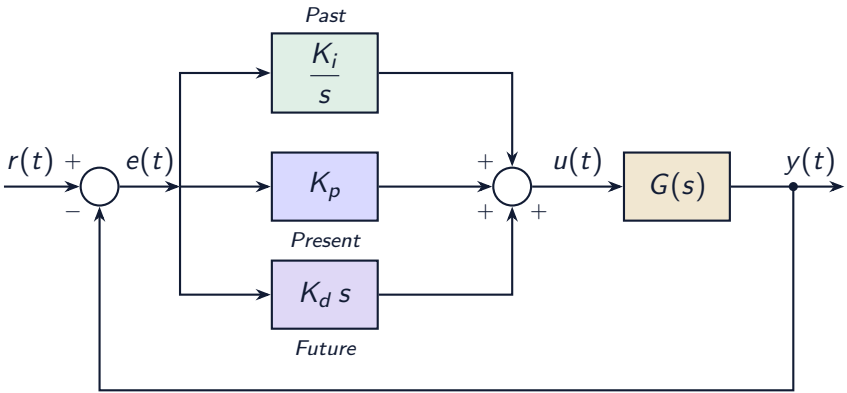
$$C(s) = K_p + \frac{K_i}{s} + K_d\, s = \frac{K_d s^2 + K_p s + K_i}{s}$$

Two things to notice already

The I term puts a *pole at the origin*. The D term's gain *grows* with s. Both facts come back on the next three slides.

Handbook § 1.5

Inside a PID controller



Proportional alone leaves an offset

Normalised speed loop $G(s) = \dfrac{1}{s+1}$, controller K_p, unit-step reference. The feedback formula gives

$$T(s) = \frac{K_p}{s + 1 + K_p} \qquad \Rightarrow \qquad y_{ss} = T(0) = \frac{K_p}{1 + K_p}, \quad e_{ss} = \frac{1}{1 + K_p}$$

$K_p = 4$: $y_{ss} = 0.8$, $e_{ss} = 0.2$ $\qquad$ $K_p = 9$: $y_{ss} = 0.9$, $e_{ss} = 0.1$

Why it can never reach zero

To hold the output up, the plant needs a non-zero u. A P controller only produces $u \neq 0$ when $e \neq 0$. More gain shrinks the offset but never removes it — and on a real plant, pushes towards oscillation.

Handbook § 1.5.1

Integral and derivative: what each buys, what each costs

Integral, K_i/s

While $e \neq 0$ the integral keeps growing. A steady state is only possible once $e = 0$ — so the offset goes. (Same plant, $K_p = 4$, $K_i = 5$: the error settles to zero.)

Costs: the pole at the origin slows and destabilises the transient; if the actuator saturates the integral *winds up*.

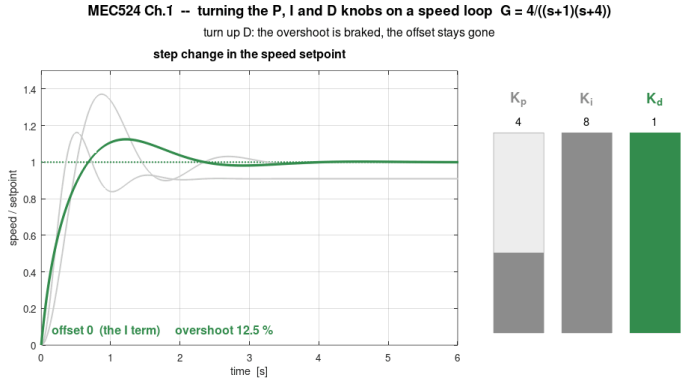
Derivative, $K_d s$

Reacts to the *rate* of change. Error closing fast? It pushes back — a brake that reduces overshoot.

Costs: at $s = j\omega$ its gain is $K_d\,\omega$, which grows without limit. Sensor noise is high-frequency, so D amplifies it. Real controllers filter it: $\dfrac{K_d s}{T_f s + 1}$.

Handbook § 1.5.2–1.5.3

Watch the three knobs, turned one at a time



A speed loop with the motor's electrical lag kept in, $G(s) = 4/[(s+1)(s+4)]$. First K_p goes up, then K_i, then K_d; grey traces are where each earlier knob left off. GIF: `ch1-anim/MEC524-ch1-pid.gif`.

What the knobs did, in numbers

Unit step in the speed setpoint, $G(s) = 4/[(s+1)(s+4)]$, unity feedback.

Knobs	Offset e_{ss}	Overshoot	What changed
$K_p = 0.5$	0.667	0%	sluggish, far short
$K_p = 4$	0.200	12.0%	closer, and it rings
$K_p = 10$	0.091	27.9%	still short, rings more
$K_p = 4,\ K_i = 8$	0	37.1%	offset gone — at a price
$K_p = 4,\ K_i = 8,\ K_d = 1$	0	12.5%	braked; settles in 2.14 s

The pattern behind the table

P alone: $e_{ss} = 1/(1 + K_p)$ is never zero. I removes the offset but adds overshoot. D brakes the overshoot. Chapter 4 finds these gains by design, not by turning knobs.

Handbook § 1.5.1–1.5.3

Tuning by hand: the ultimate-gain experiment

Ziegler–Nichols, closed loop

1. Switch off I and D: $K_i = K_d = 0$.
2. Raise K_p until the output oscillates with *constant* amplitude. Record that gain, K_u, and the period, P_u.
3. Read the gains from the table.

Controller	K_p	K_i	K_d
P	$0.5\,K_u$	0	0
PI	$0.45\,K_u$	$0.54\,K_u/P_u$	0
PID	$0.6\,K_u$	$1.2\,K_u/P_u$	$0.075\,K_uP_u$

Behind the PID row: integral time $T_i = K_p/K_i = P_u/2$, derivative time $T_d = K_d/K_p = P_u/8$.

Handbook § 1.5.4–1.5.5, Table 1.4

Worked — tuning a DC-motor speed loop (1 of 2)

A bump test on a DC-motor speed loop, P only: at $K_p = 18$ the speed oscillates steadily with period 0.8 s. Choose PID gains.

Step 1 — name what was measured. The gain at sustained oscillation is the ultimate gain; its period is the ultimate period:

$$K_u = 18, \qquad P_u = 0.8 \text{ s}$$

Step 2 — pick the row. A full PID is asked for: bottom row.

Step 3 — arithmetic.

$$K_p = 0.6 \times 18 = 10.8, \qquad K_i = \frac{1.2 \times 18}{0.8} = 27 \text{ s}^{-1}, \qquad K_d = 0.075 \times 18 \times 0.8 = 1.08 \text{ s}$$

Handbook § 1.6.4, Problem 14

Step 4 — check through the time constants.

$$T_i = \frac{K_p}{K_i} = \frac{10.8}{27} = 0.4\ \mathrm{s} = \frac{P_u}{2} \qquad T_d = \frac{K_d}{K_p} = \frac{1.08}{10.8} = 0.1\ \mathrm{s} = \frac{P_u}{8}$$

Step 5 — treat the result as a starting point. Ziegler–Nichols gains are deliberately aggressive: fast, with visible overshoot. A typical first refinement trims K_p and K_i by about 30%:

$$K_p \approx 7.56, \qquad K_i \approx 18.9\ \mathrm{s}^{-1}$$

then confirms on a step test.

Where this goes

Chapter 4 places the gains with the root locus; Chapter 5 with gain and phase margins. Z–N is the bench method you use before you have a model.

Handbook § 1.6.4, Problem 14

Three misreadings the animations should cure

1. **"Closed loop means perfect."**
   With P control the turntable still ends at 32.9, not 33.3 rev/min. Feedback shrinks an error; only an integrator removes it.

2. **"Enough K_p kills the offset."**
   $K_p = 10$ still leaves 0.091 — and 27.9% overshoot. $1/(1 + K_p)$ is zero only at infinite gain.

3. **"A pole near the axis is safe because it is still on the left."**
   It is stable, but slow: at $\sigma = -0.1$ the envelope needs $4/0.1 = 40$ s to fall under 2%. Stable and *acceptable* are different questions.

Three ways to lose marks here

1. **Transforming a product piece by piece.**
   $\mathcal{L}\{e^{2t}\cosh t\}$ is an s-shift, not $\frac{1}{s-2}\cdot\frac{s}{s^2-1}$. Products in time are not products in s.

2. **Using cover-up on every coefficient of a repeated pole.**
   It only gives the highest power. The rest need a derivative, or matching powers of s.

3. **Skipping the free checks.**
   $x(0)$, $\dot{x}(0)$, and the simple-pole coefficients summing to zero each take one line — and catch a wrong coefficient before the marker does.

And one from the concepts

Calling a system closed loop because it has a controller. Closed loop means the *output is measured and fed back*.

What the assessment will ask

Assessment	Week	Weight	What it asks from this chapter
Quiz 1	Wk 4	5% CLO1	Laplace pairs and properties, inverse transforms by partial fractions, and open- versus closed-loop concepts.
Test 1	Wk 9	30% CLO1+CLO2	Transforming an ODE, inverting by partial fractions, a transfer function and its poles, and basic PID reasoning.

What a full-mark answer looks like

- the pole types named *before* the expansion is written;
- each coefficient found by a stated method (cover-up, derivative, matching);
- one line of check against the initial conditions;
- the time answer stated for $t \geq 0$, with units where the problem has them.

What to take away

1. Laplace turns d/dt into s and $\int dt$ into $1/s$: an ODE becomes algebra.
2. Invert by sorting poles — distinct, repeated, complex — then use the table.
3. $G(s)$ describes the plant alone; poles in the left half-plane mean stable.
4. Feedback, $T = G/(1 + GH)$, buys insensitivity and disturbance rejection and pays in dynamics and noise.
5. P reacts, I removes offset, D damps; Ziegler–Nichols gives a starting point.

Handbook § 1.0–1.6. Worked examples: § 1.2 and § 1.6.
Tutorial: § 1.6 — attempt it before the tutorial class.