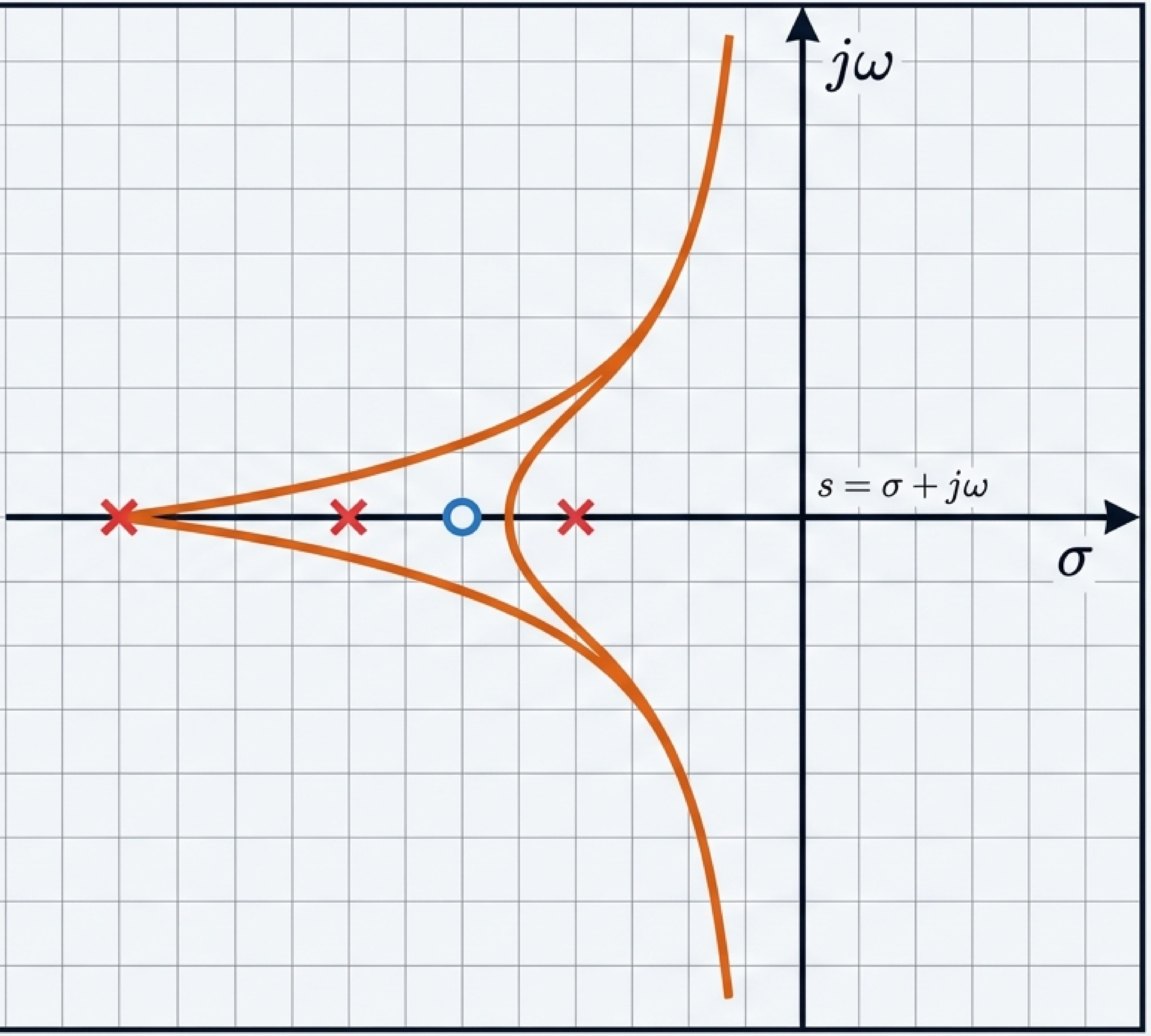
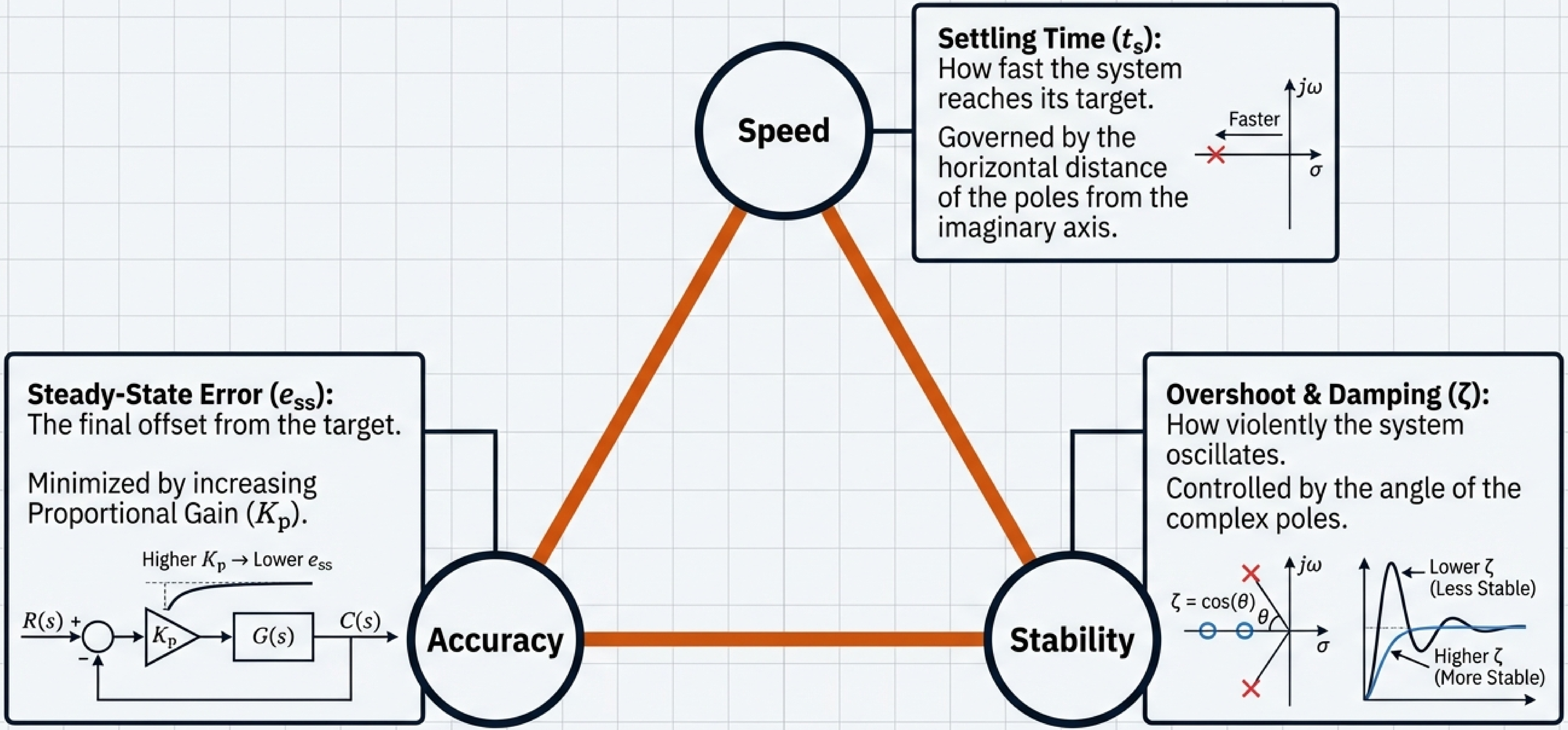
# Mapping System Stability

A Visual Guide to Root Locus & Control Performance.






**Key Takeaway:** You rarely get all three simultaneously. The Root Locus is our mathematical map for finding the perfect compromise.

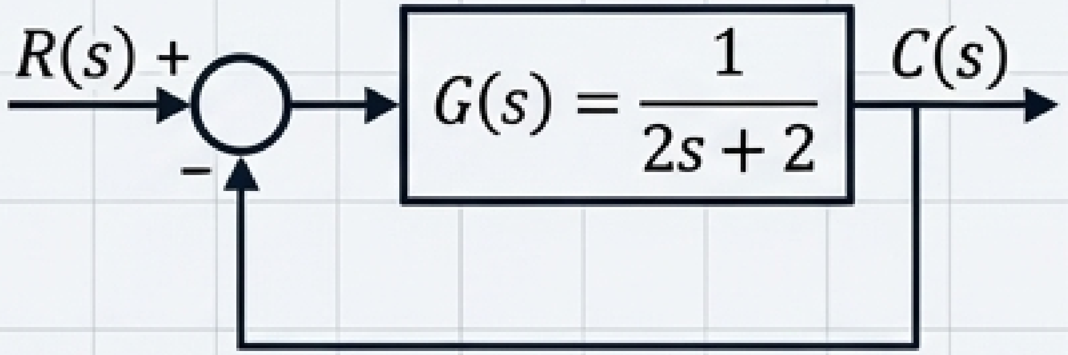
# Anatomy of System Dynamics

## 1st Order System

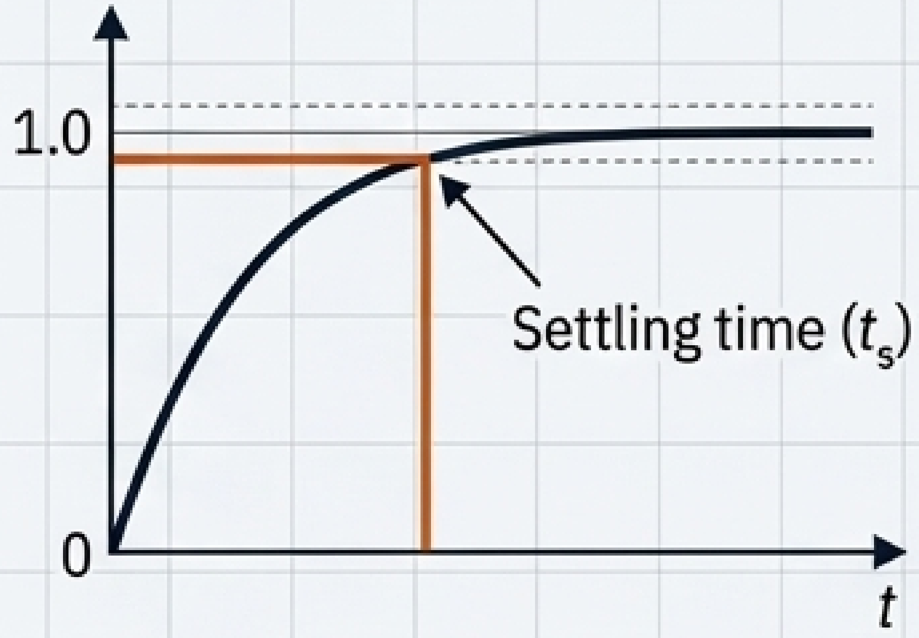
**The Math**

$$G(s) = \frac{1}{2s+2}$$

**System Diagram**




**Physical Reality**

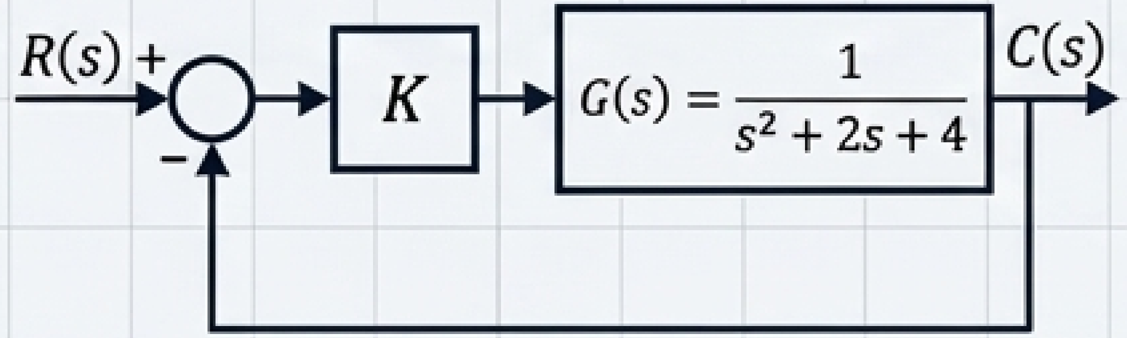



## 2nd Order System

**The Math**

$$G(s) = \frac{K}{s^2+2s+4}$$

**System Diagram**

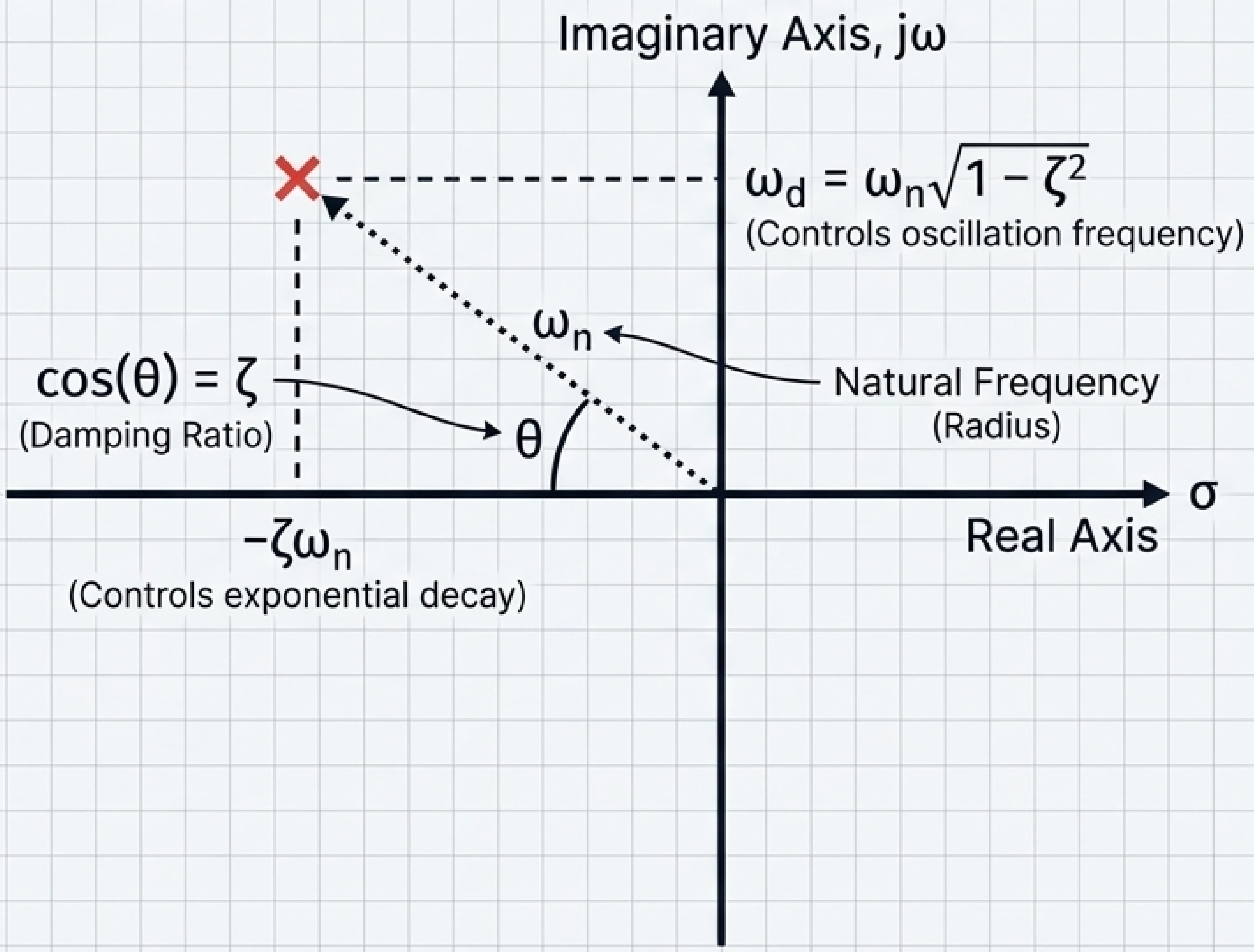



**Physical Reality**

# Defining the Limits: Geometry of the *s*-Plane

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

The physical reality of a machine is entirely dictated by where its mathematical roots "live" on this map.



# The Role of the Controller

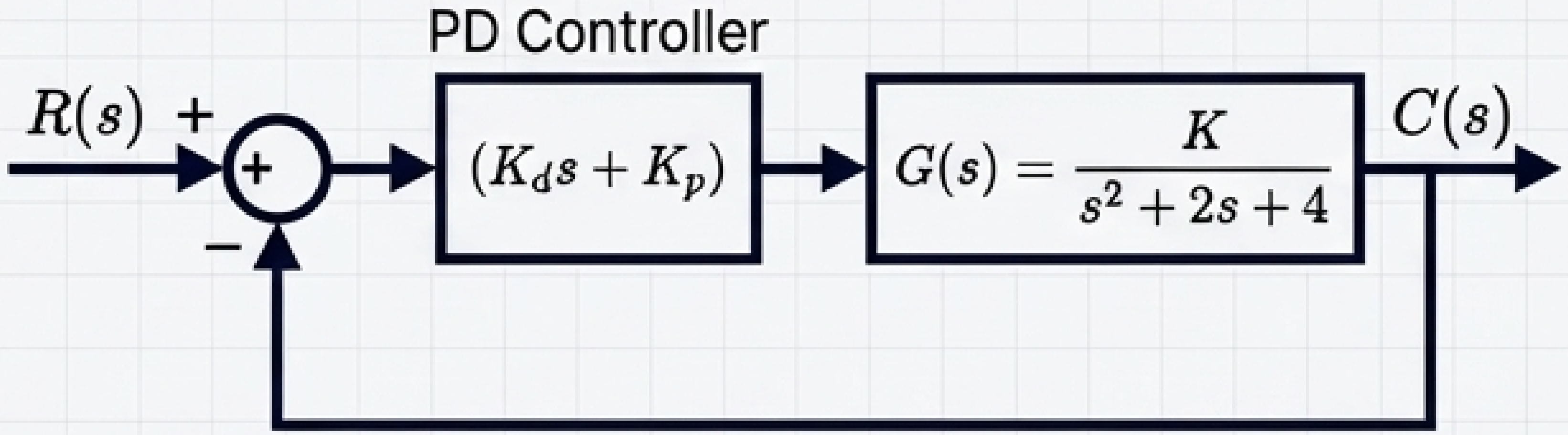
1. **The Requirement:** Achieve 93.8% accuracy (Steady-State Error $e_{ss} = 0.062$) to a unit step input.

2. **The Calculation:**

$$e_{ss} = \lim_{s \to 0} s \left( \frac{1}{s} \frac{1}{1 + K_p/4} \right) = 0.062$$

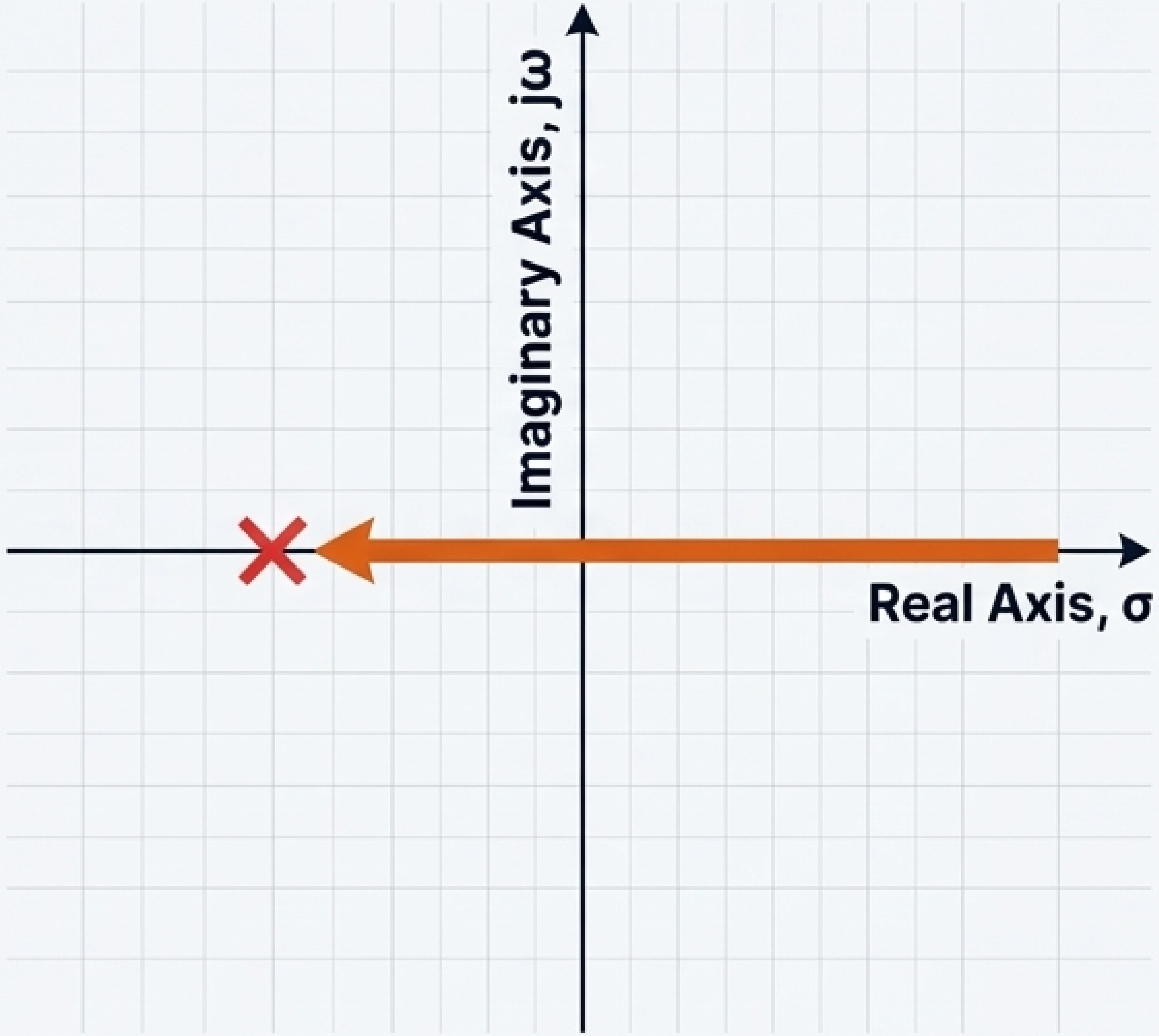
3. **The Result:** Proportional Gain $K_p = 60.52$

4. **The Tuning:** Matching the characteristic equation yields Damping Gain $K_d = 13.359$ to maintain $\zeta = \frac{1}{\sqrt{2}}$



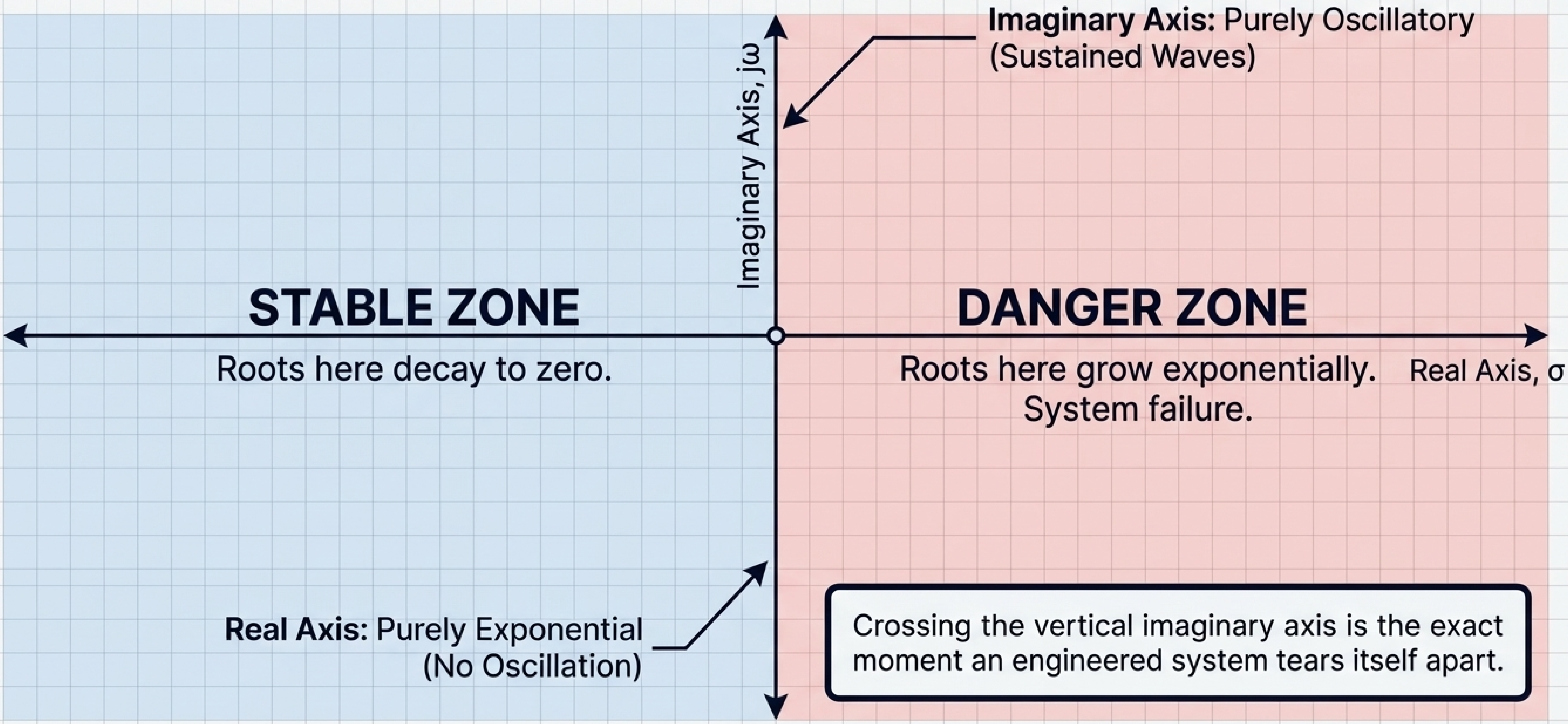

A controller forces a system's roots to relocate to a desired, stable coordinate on the s-plane.

# What is the Root Locus?


**If Gain (K) is a volume knob from zero to infinity, the Root Locus is the visual track of how the system's roots migrate in response. As K increases, the physical capabilities of the machine change. We must map their path before we turn up the power.**

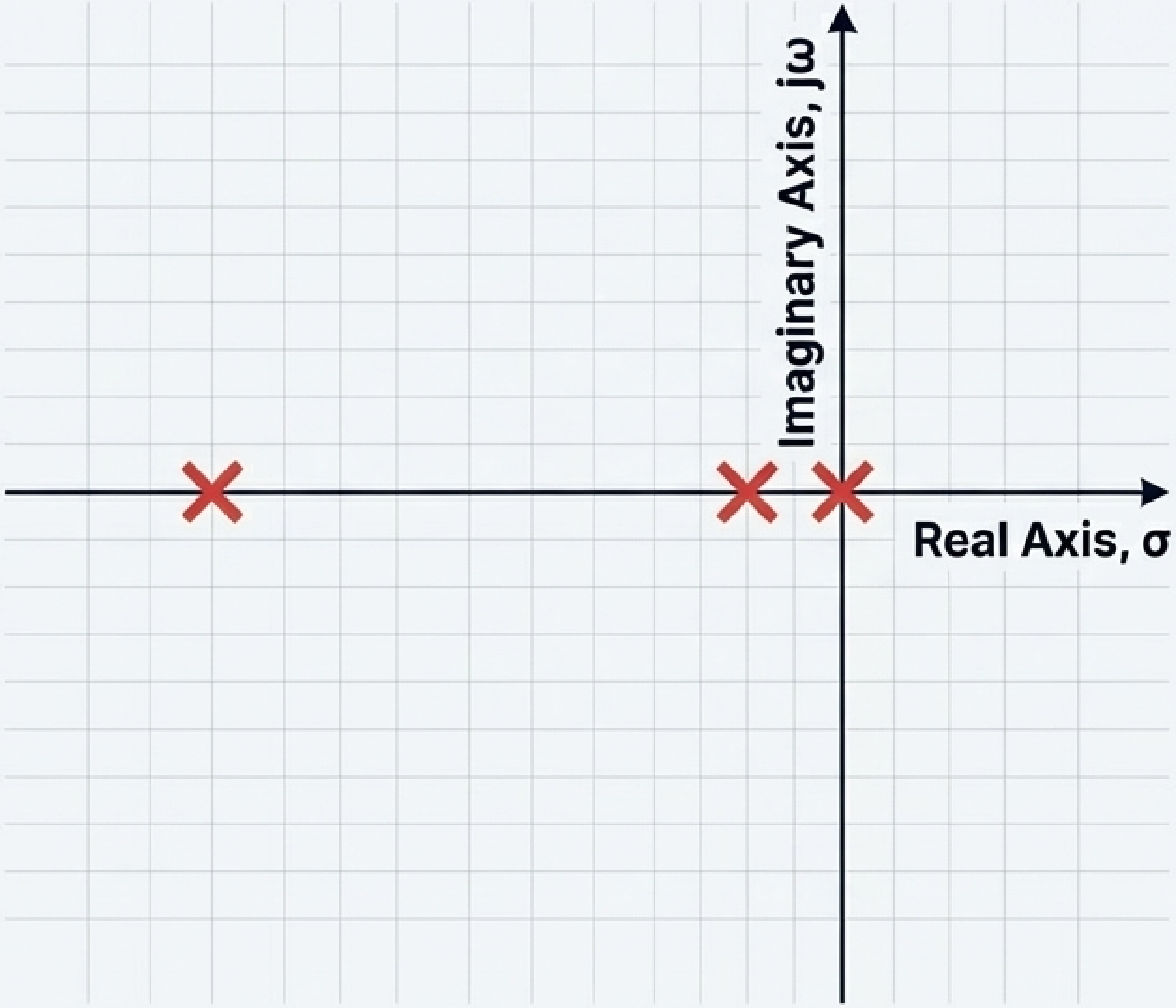
# Reading the Map: The S-Plane Zones




Crossing the vertical imaginary axis is the exact moment an engineered system tears itself apart.

# Step 1 & 2: Finding the Origins

## The 7-Step Locus Blueprint




**Step 1: The Origins (Poles, n)**

Roots of the open-loop denominator. This is where the path begins at K=0.

Example: $s_1 = 0$, $s_2 = -1$, $s_3 = -10$ (n=3)

**Step 2: The Destinations (Zeros, m)**

Roots of the open-loop numerator. This is where the path ends at K=∞.

In this example, m=0.

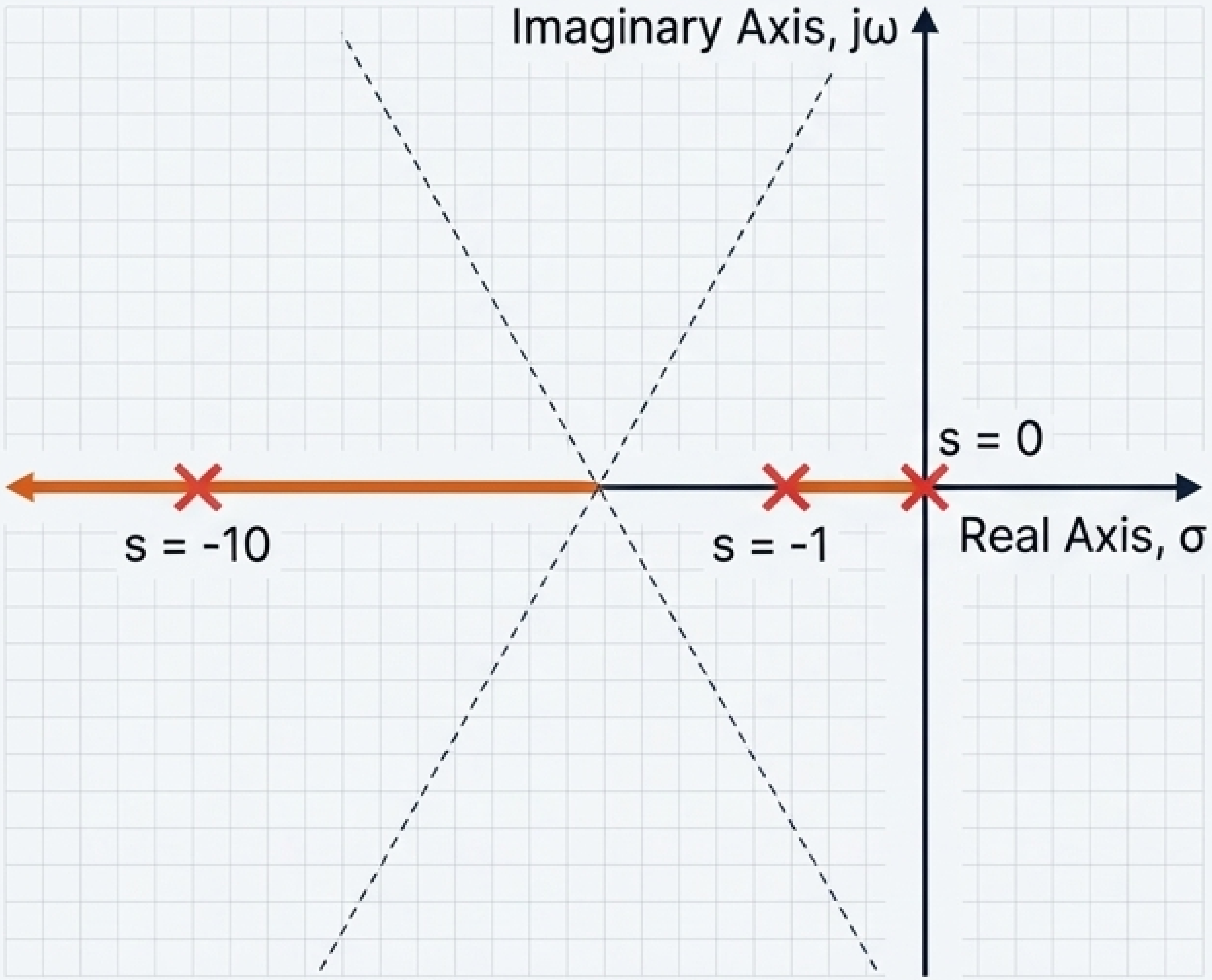
**The Rule of Branches:**

The number of branches that travel off the map to infinity is exactly $n - m$.

Example: $3 - 0$ = 3 branches.

# Step 3 & 4: The Guides

The 7-Step Locus Blueprint




## Step 3: The Guides (Asymptotes)

Where do the infinite branches go?

Centroid: $\sigma_a = \frac{\sum \text{Poles} - \sum \text{Zeros}}{n - m} = \frac{-11}{3} = -3.66$

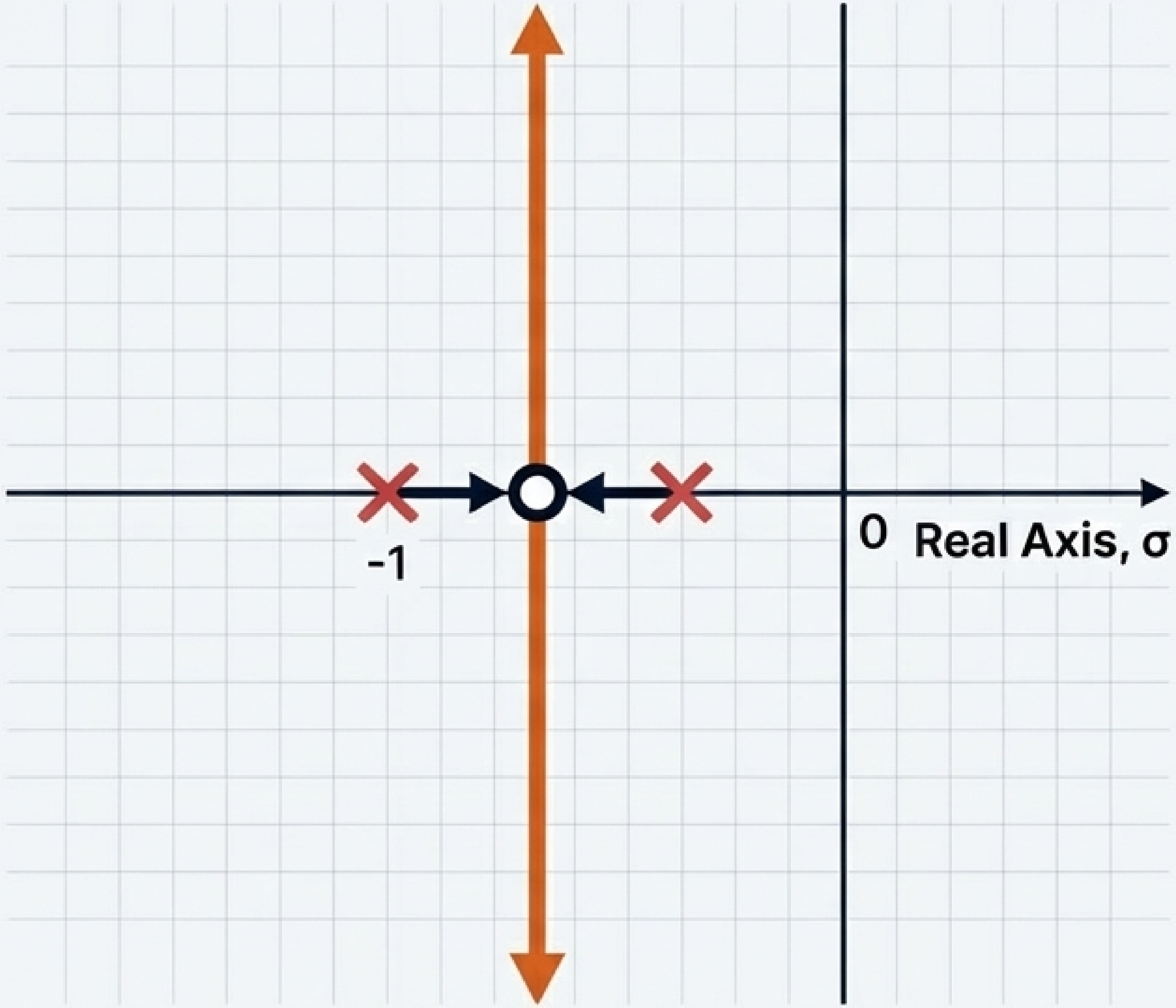
Angles: $\phi_a = \frac{(2q+1)180}{n-m} \rightarrow 60°, 180°, 300°$

## Step 4: The Real Locus

The Odd Rule of Thumb: Stand on the real axis and look **right**. If you see an ODD number of poles/zeros, you are standing on the locus track. Draw a line.

# Step 5: The Collision

The 7-Step Locus Blueprint




**Step 5: The Collision (Breakaway Points)**

When two roots meet head-on, they break away into the complex plane, introducing oscillation.

**The Math:** Take the derivative of the characteristic equation with respect to s and set it to zero: $\frac{dK}{ds} = 0$.

**The Calculation:**

$$K = -(0.1s^3 + 1.1s^2 + s)$$

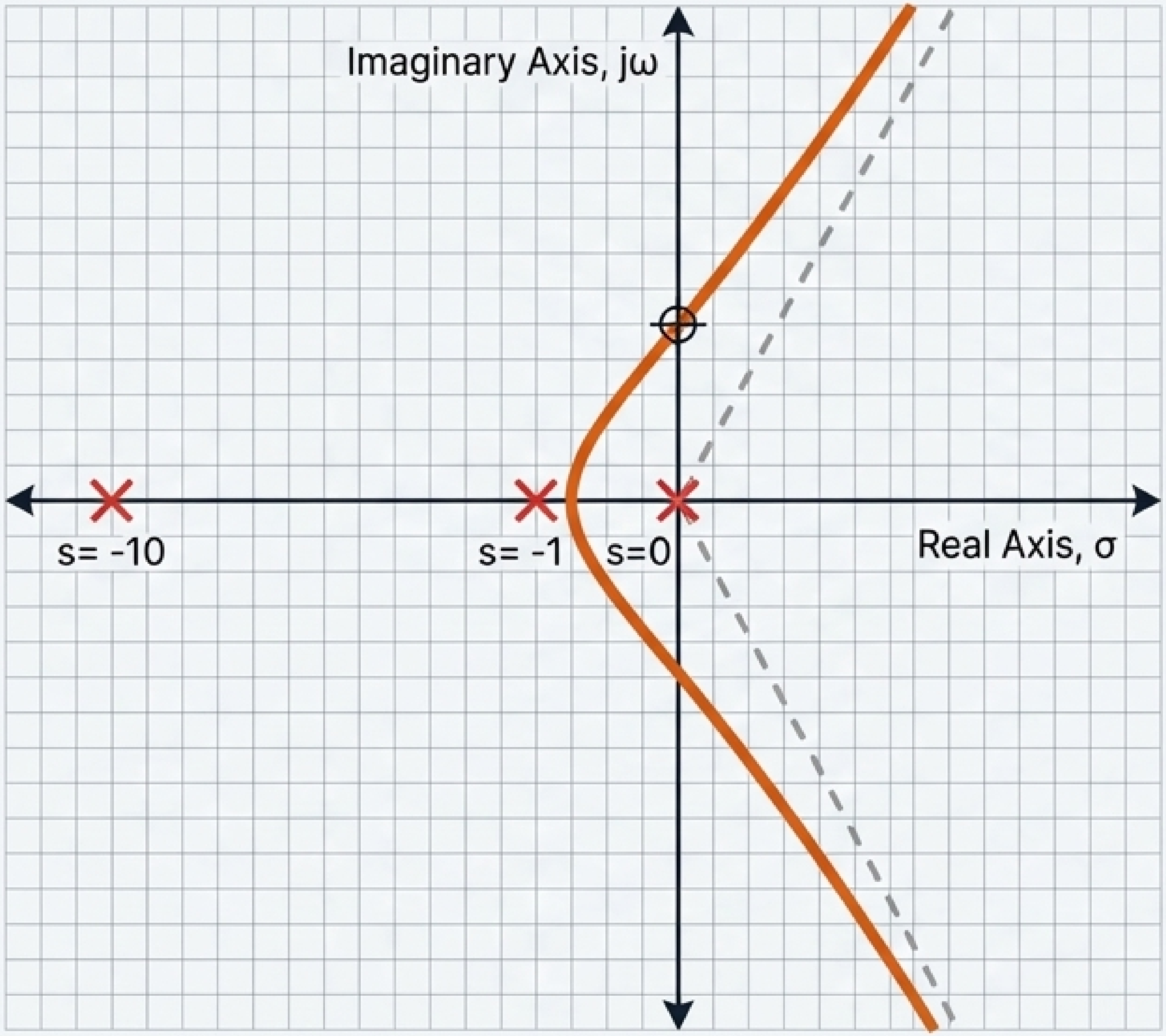
$$\frac{dK}{ds} = -(0.3s^2 + 2.2s + 1) = 0$$

**The Coordinate:**

The exact pixel of breakaway occurs at s = -0.486.

# Step 6 & 7: The Danger Zone

The 7-Step Locus Blueprint




## Step 6: Cross-over Frequency

Find the exact point where the track crosses the vertical axis ($s = j\omega$).

Substitute $s = j\omega$ into the equation:

$$-1.1\omega^2 + K + j\omega(1 - 0.1\omega^2) = 0$$

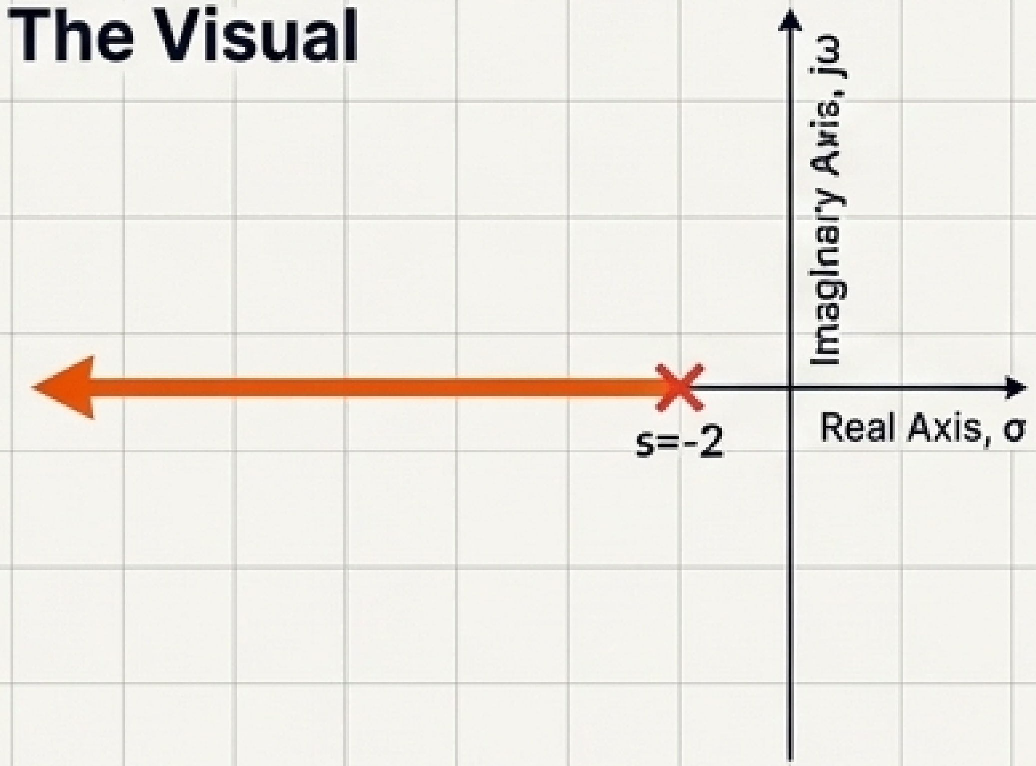
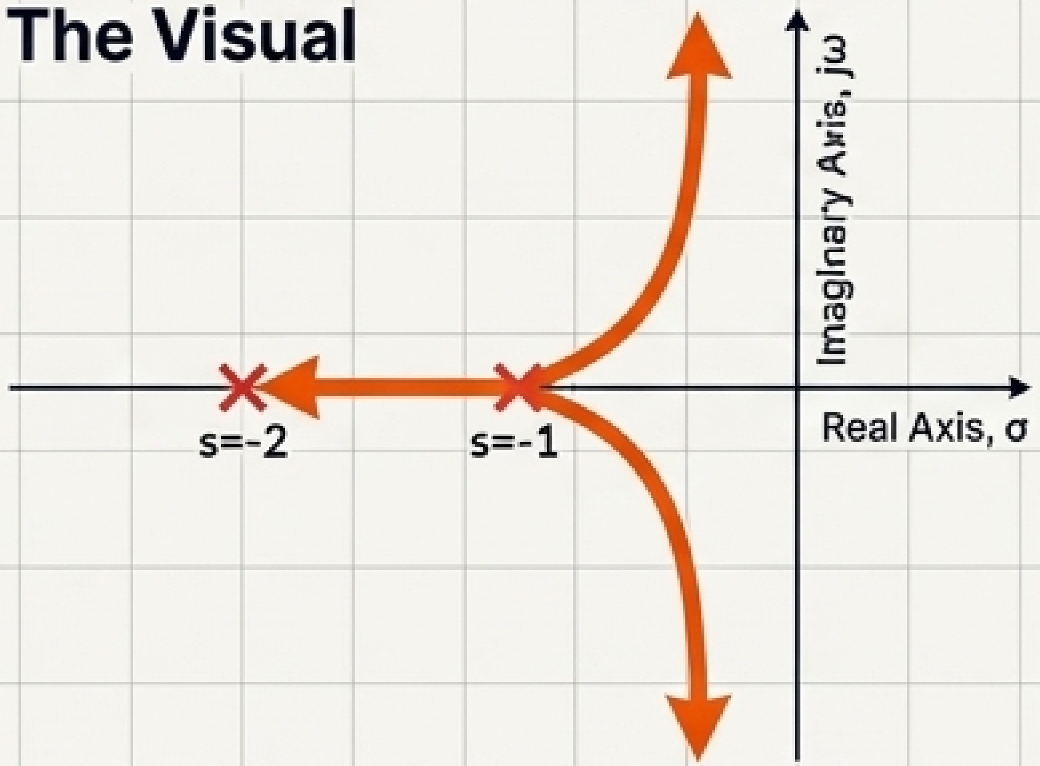
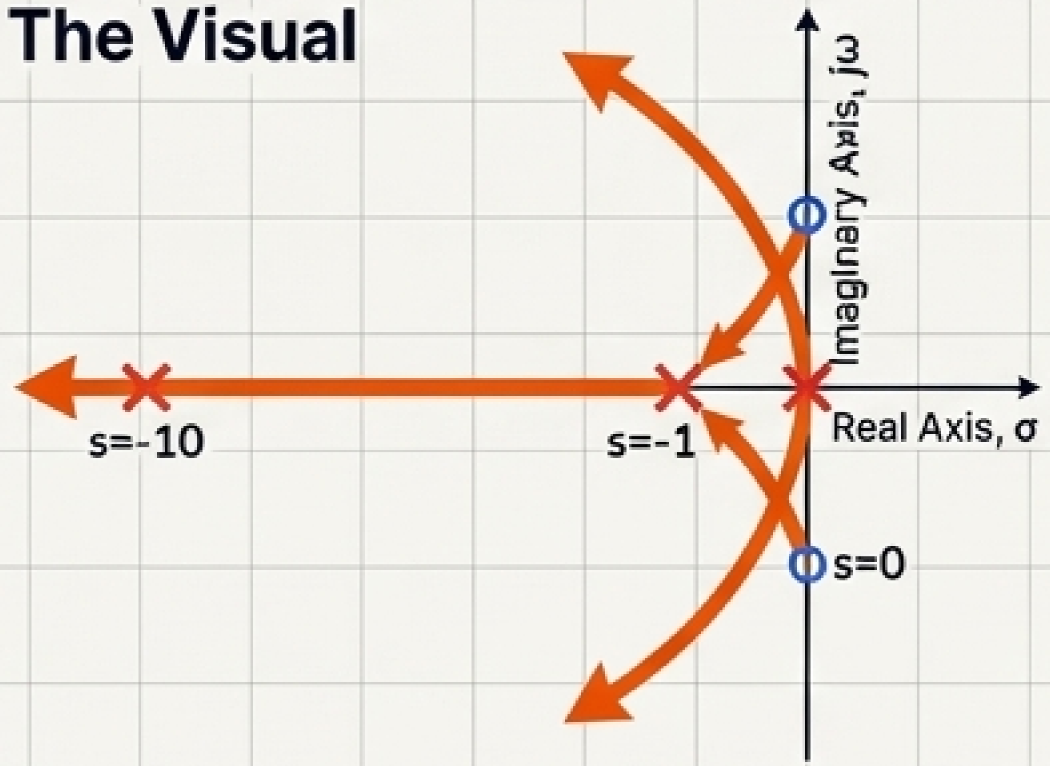
**The Threshold of Failure:**

The locus crosses at critical frequency $\omega = \sqrt{10}$ rad/s.
At this exact boundary, the Critical Gain $K_{\max} = 11$.

## Step 7: Departure Angles

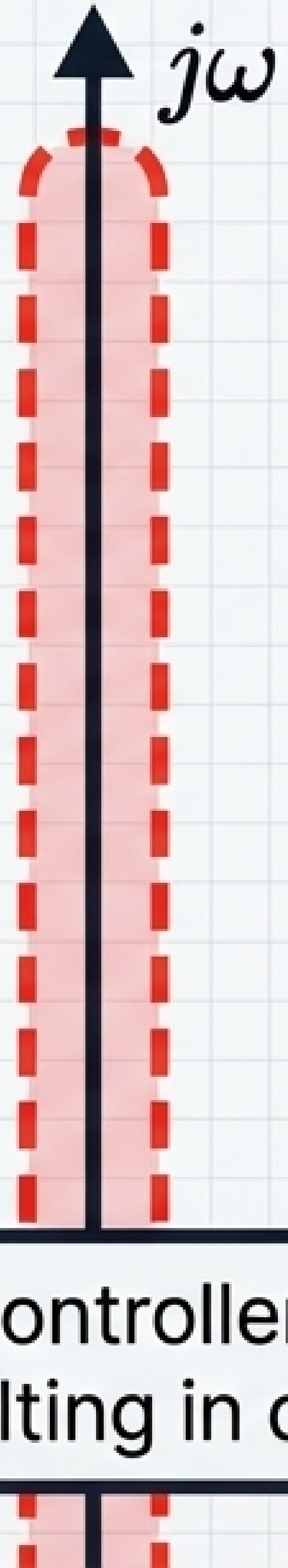
**Note:** Only applies when starting with complex roots. Calculate the initial trajectory angle using $\sum\theta_{\text{zeros}} - \sum\theta_{\text{poles}}$.

# The Matrix: How Architecture Shapes the Path

| | 1-Pole System | 2-Pole System | 3-Pole System |
|---|---|---|---|
| **The Math** | $G(s) = \dfrac{K}{s+2}$ | $G(s) = \dfrac{K}{(s+1)(s+2)}$ | $G(s) = \dfrac{K}{s(0.1s+1)(s+1)}$ |
| **The Visual** |  |  |  |
| **The Implication** | Always stable. Becomes purely faster as K increases. | Always stable. Becomes infinitely oscillatory but never fails. | Conditionally stable. If K is turned too high (K > 11), the system breaks. |

# Finding the Maximum Gain ($K_{\max}$)

When does the system break?




1. Define the System Equation

$$s^3 + 12s^2 + 44s + 48 + K = 0$$

2. Set $s = j\omega$ to target the intersection:

Real part: $K + 48 - 12\omega^2 = 0$

Imaginary part: $\omega(44 - \omega^2) = 0$

3. Solve for the Frequency Boundary

$$\omega = \sqrt{44}\,\text{rad/s}$$

4. Substitute back to find Critical Gain

$$K_{\max} = 12(44) - 48$$

$$K_{\max} = 480$$

**The Absolute Limit:** Turning the controller dial past $K = 480$ pushes the roots over the boundary, resulting in catastrophic system failure.

# Engineering for Specifics

**1. The Business Requirement**

The real-world machine must operate with exactly 15% overshoot.

**2. The Geometry**

Translate the requirement into a Damping Ratio ($\zeta$):

$$\zeta = \frac{-\ln(0.15)}{\sqrt{\ln(0.15)^2 + \pi^2}}$$

$$= 0.591$$

**3. The Coordinates**

Calculate exactly where the roots must live on the s-plane:

$$s = -2.02 \pm 2.79j$$

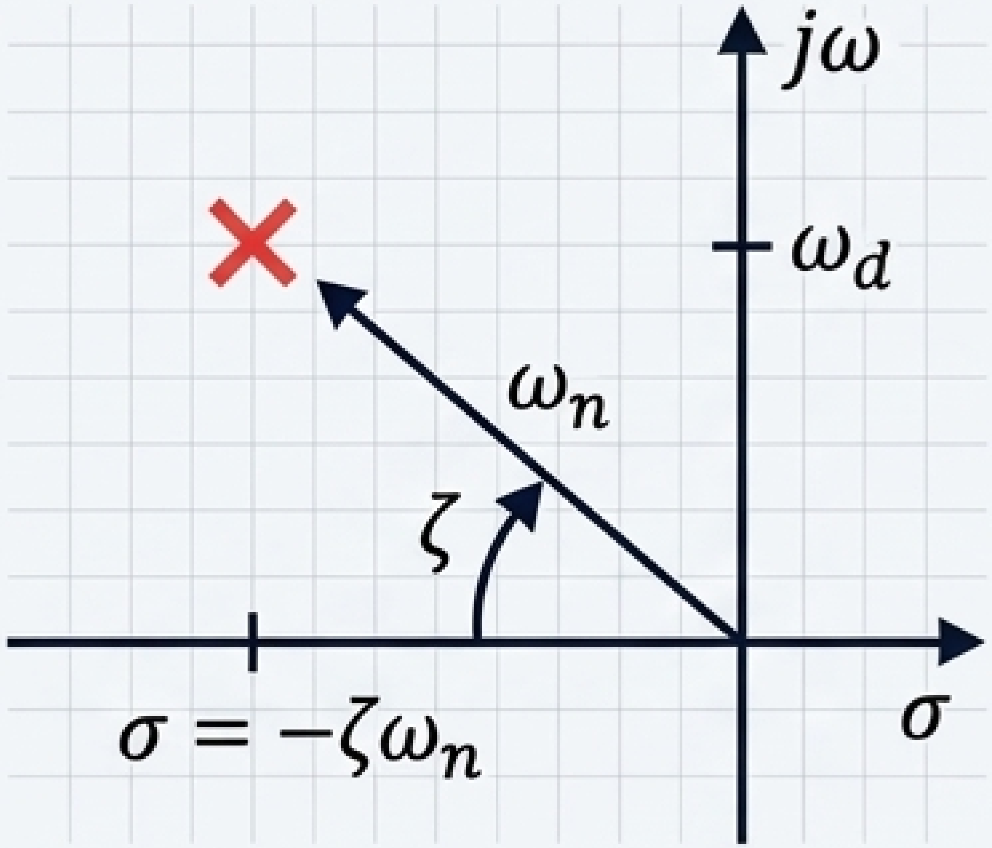
**4. The Engineering Parameter**

Solve the characteristic equation to find the exact dial setting to hit that pixel:

$$K = 45.88$$

# The Engineer's Quick-Reference Cheat Sheet

## The Geometry of Roots




2nd Order Standard Form:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

## The 7 Rules of Root Locus

1. Origins: Poles ($K$=0)
2. Ends: Zeros ($K$=∞)
3. Asymptotes: $\sigma_a, \phi_a$
4. Real Axis: Odd-rule
5. Breakaway: $dK/ds = 0$
6. Axis Crossing: $s = j\omega$
7. Departure Angles: $\sum\theta_Z - \sum\theta_P$

## Map Legend

= System Pole

= System Zero

= Locus Track (Path of Roots)

= Stable Zone (LHP)

= Unstable Zone (RHP)