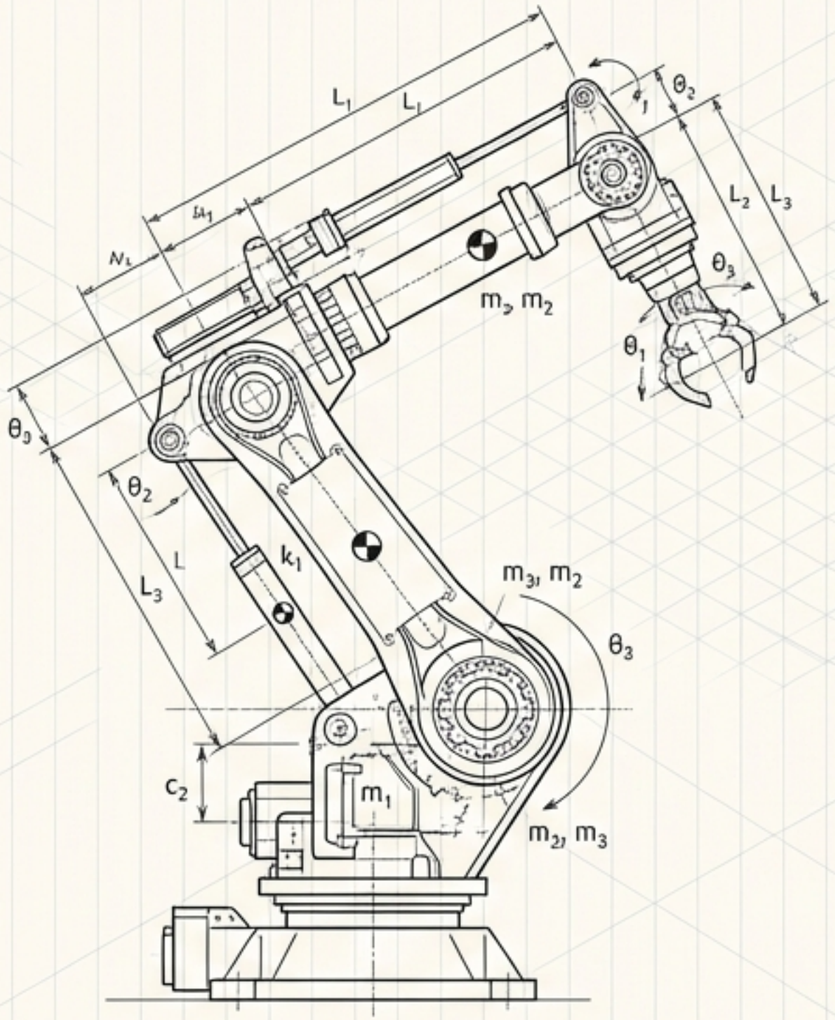
From Physics to Matrices: The Control Engineer's Blueprint

Translating Mechanical Dynamics into Mathematical Models



$$[M] = \begin{bmatrix} (m_1+m_2+m_3)L_1^2 + J_1 + J_3 & (m_2+m_3)L_1L_2\cos(\theta_2-\theta_1) & m_3L_1L_3\cos(\theta_3-\theta_1) \\ (m_2+m_3)L_1L_2\cos(\theta_2-\theta_1) & (m_2+m_3)L_2^2 + J_2 + J_3 & m_3L_2L_3\cos(\theta_3-\theta_2) \\ m_3L_1L_3\cos(\theta_3-\theta_1) & m_3L_2L_3\cos(\theta_3-\theta_2) & m_3L_3^2 + J_3 \end{bmatrix}$$

$$+ [C] = \begin{bmatrix} c_1 + c_2 + c_3 & -c_2 & -c_3 \\ -c_2 & c_2 + c_3 & -c_3 \\ -c_3 & -c_3 & c_3 \end{bmatrix} [\dot{\theta}]$$

$$+ [K] = \begin{bmatrix} k_1 + k_2 + k_3 & -k_2 & -k_3 \\ -k_2 & k_2 + k_3 & -k_3 \\ -k_3 & -k_3 & k_3 \end{bmatrix} [\theta] = [F]$$

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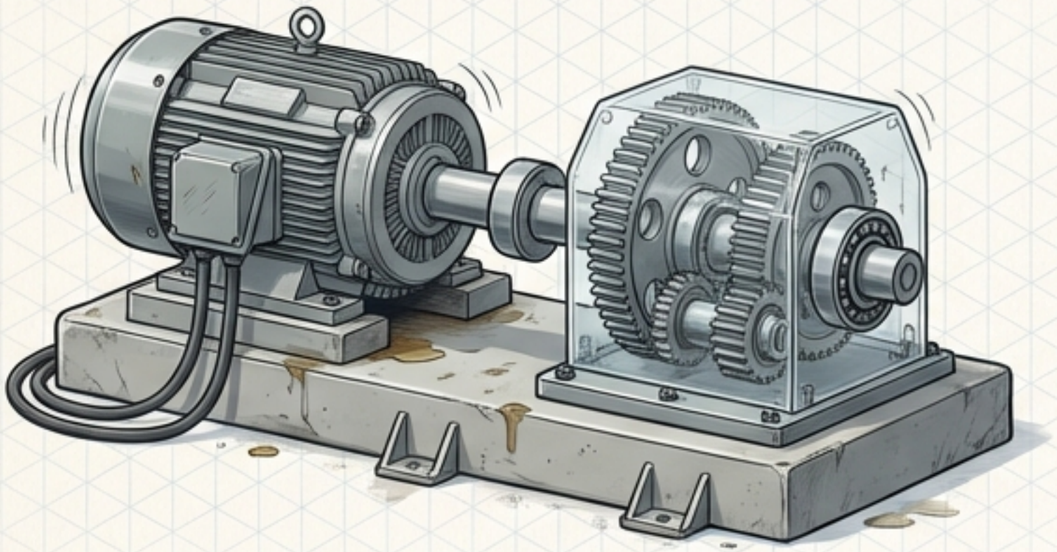
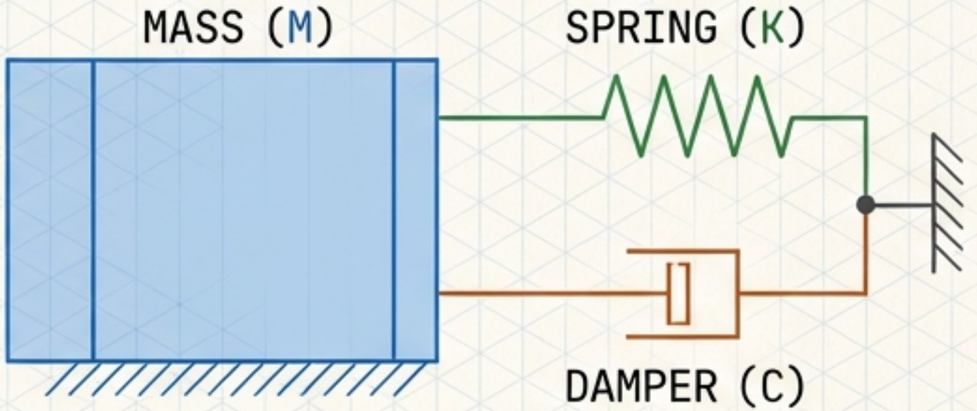
The Map is Not the Territory

THE TERRITORY



THE MAP



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PHYSICAL REALITY

Physical systems are infinitely complex. Analyzing their raw, unpredictable dynamics in the real world is risky, expensive, and subject to environmental noise.

MATHEMATICAL MODEL

A mathematical model is a deliberate simplification. Like a map, it strips away physical noise to capture only the essential dynamic relationships required for navigation, analysis, and control.

Mathematical Modelling: Deriving equations from fundamental physical laws (Newton, Kirchhoff).

System Identification: Constructing models empirically from measured input-output experimental data.

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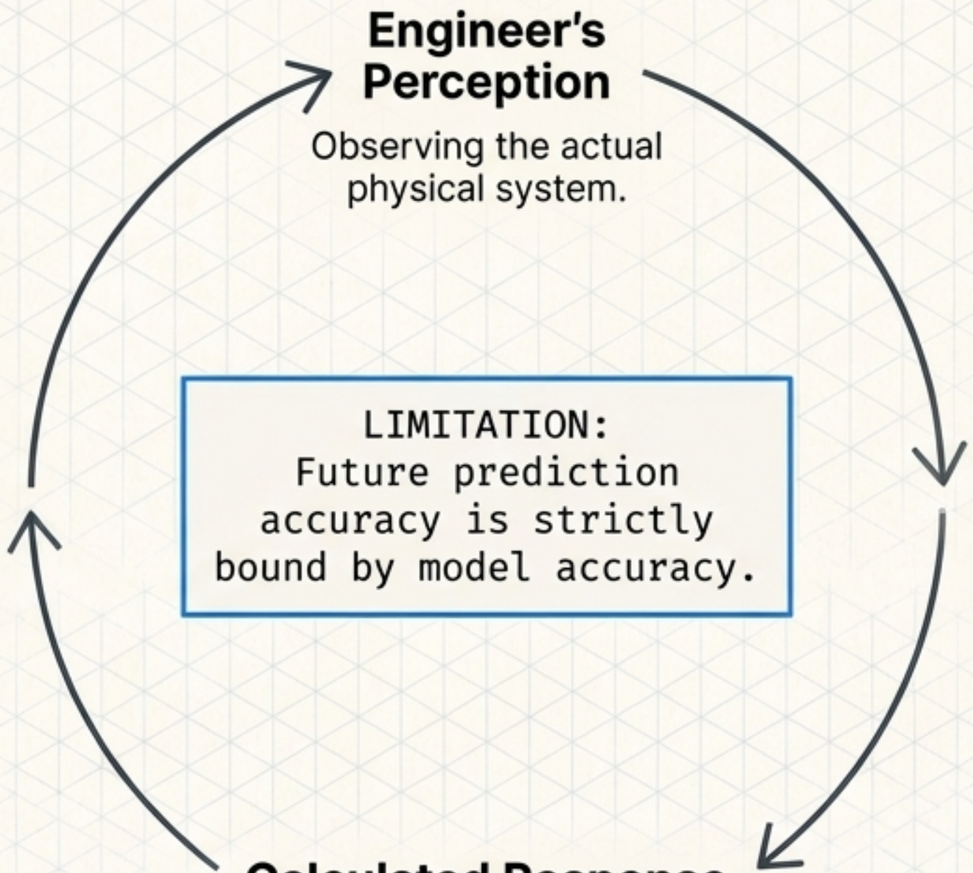
The Virtual Prototyping Loop



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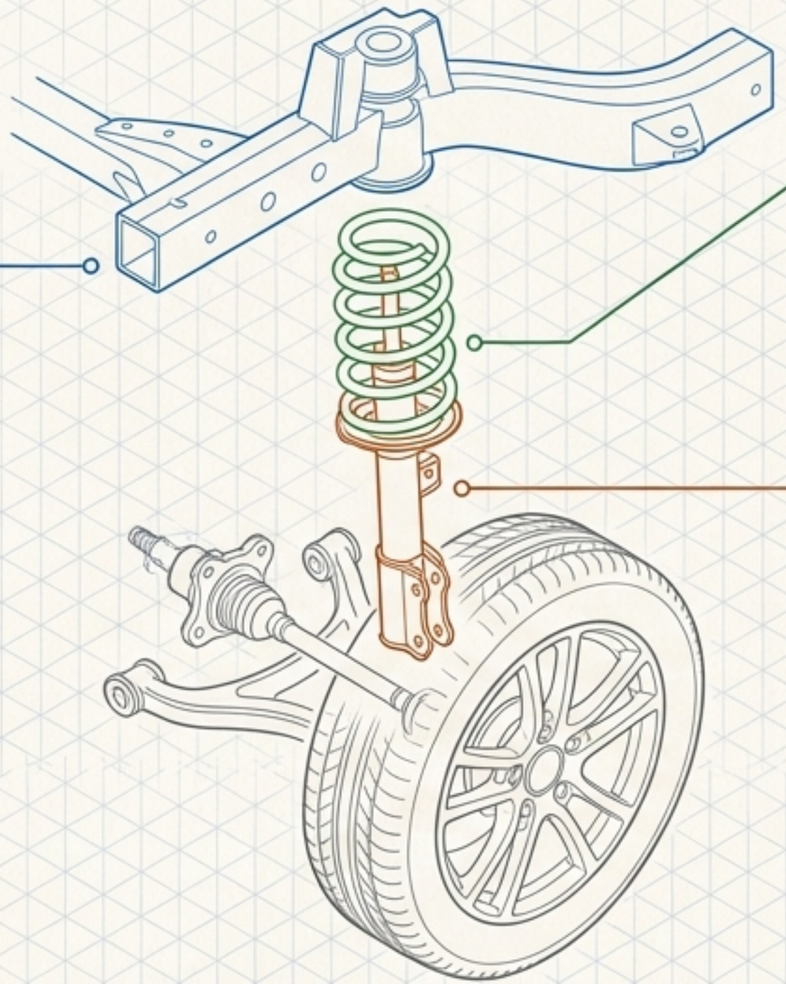
The Anatomy of a Dynamic System



The Car Body (Mass)

- **Function:** Stores Kinetic Energy.
- **Role:** The inertial resistance to acceleration.

The Suspension Spring (Spring)

- **Function:** Stores & Returns Potential Energy.
- **Role:** Creates bouncing motion; force is proportional to deflection.

The Shock Absorber (Damper)

- **Function:** Dissipates Energy as Heat.
- **Role:** Prevents continuous bouncing; generates force proportional to velocity difference.

Without damping, hitting a bump would cause the mass to bounce indefinitely on the spring. The interplay of these three fundamental elements dictates the entire system response.

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The Mechanical Element Dictionary

Element	Physical Function	Translational Law (Force)	Rotational Law (Torque)
Mass / Inertia	Resists motion changes.	$F = m(d^2\dot{x}/dt^{\ddot{2}})$	$M = I(d^2\ddot{\theta}/dt^{\ddot{2}})$ where $I_O = I_G + md^2$
Spring	Deflection resistance.	$F = k(x_1 - x_2)$	$T = k_t(\theta_1 - \theta_2)$
Damper	Velocity resistance.	$F = c\left(\frac{d\dot{x}_1}{dt} - \frac{d\dot{x}_2}{dt}\right)$	$T = c_t\left(\frac{d\dot{\theta}_1}{dt} - \frac{d\dot{\theta}_2}{dt}\right)$

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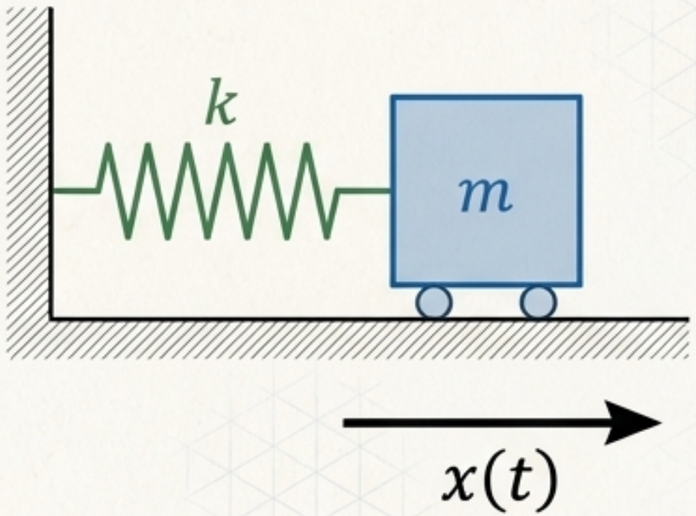
Defining the Coordinate Space (Degrees of Freedom)

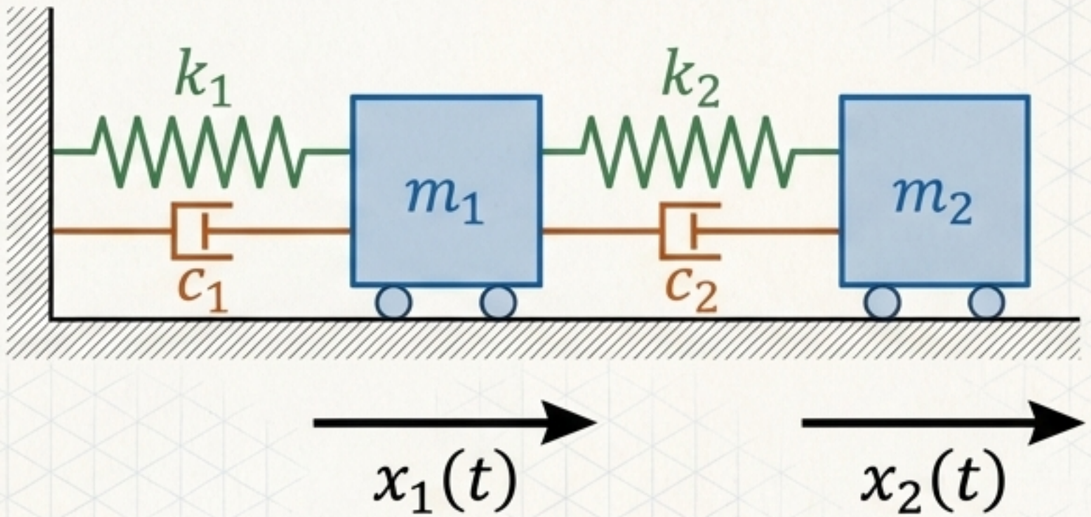
Degree of Freedom (DOF) = The number of independent generalized coordinates required to completely specify the configuration of a system.

1-DOF System



Requires only 1 coordinate to know exactly where the entire system is.

2-DOF System



The position of m_2 cannot be known merely by knowing m_1 Two independent coordinates are strictly required.

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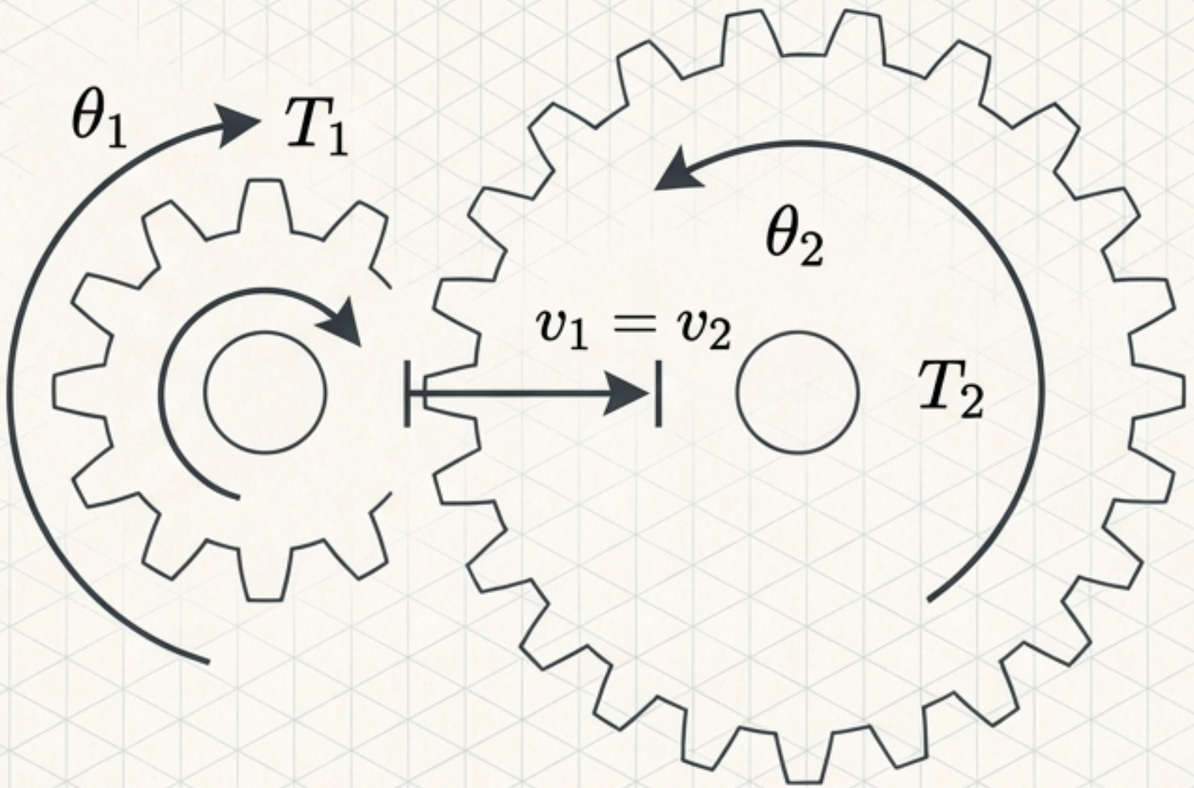
Rotational Transmission: Gear Logic



Speed Ratio (n)

$$n = \frac{\theta_1}{\theta_2} = \frac{T_2}{T_1}$$

The Golden Rule

The larger gear rotates slower but outputs proportionally higher torque.

Equivalent Inertia Translation

$$J_{eq} = J_1 + \frac{J_2}{n^2}$$

Shows how a secondary inertia is ‘felt’ by the primary shaft, scaled inversely by the square of the speed ratio.

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The Lenses of Dynamics: Four Mathematical Architectures

Configuration Form	State-Space Representation	Input-Output Equation	Transfer Function
$M\frac{d^2q}{dt^2} + C\frac{dq}{dt} + Kq = F$ (Matrix Form)	$\frac{dx}{dt} = Ax + Bu$ (State) $y = Cx + Du$ (Output)	Single n-th order differential equation.	$G(s) = \frac{Y(s)}{U(s)}$
Standard second-order matrix. Directly maps to physical geometry.	First-order differential equations. Ideal for multi-input, multi-output (MIMO) modern control theory.	Classical time-domain analysis.	Laplace domain. Assumes zero initial conditions. Perfect for frequency-domain analysis.

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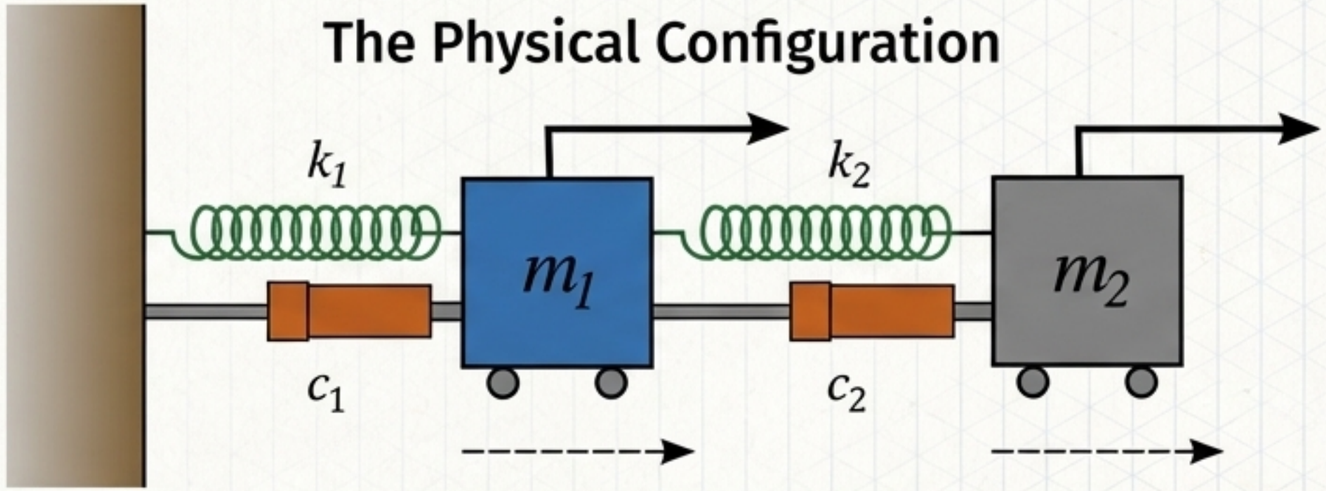
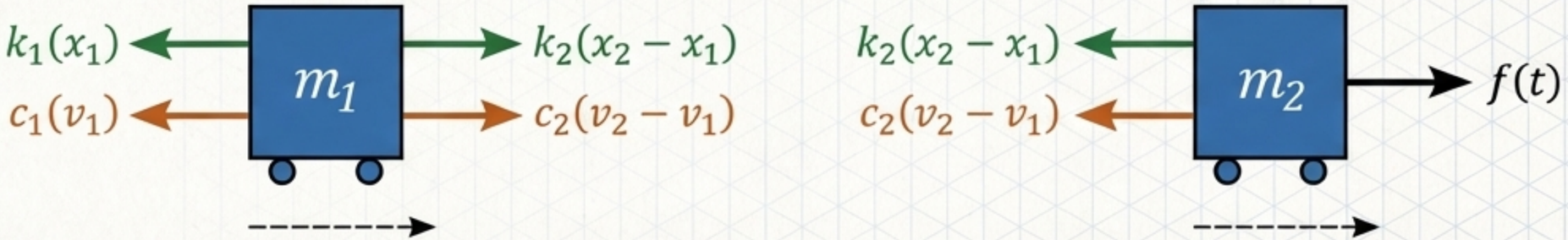
Constructing the Blueprint: Step 1 (Isolation)

The Physical Configuration



Apply Newton's 2nd Law & Isolate Elements

The Free Body Diagram (FBD)



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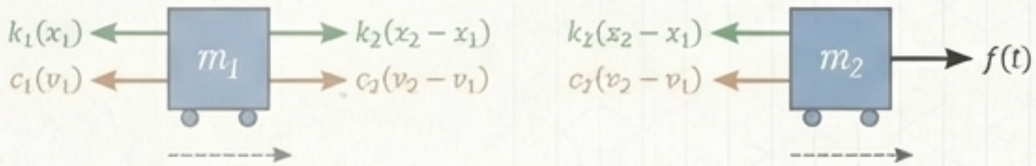
Constructing the Blueprint: Step 2 (Equations of Motion)

Reference Context



The Math

$$m_1\ddot{x}_1 = k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) - k_1x_1 - c_1\dot{x}_1$$

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) + f(t)$$

Key Insight: Every arrow on the Free Body Diagram becomes exactly one term in the differential equations. Notice how relative motion $(x_2 - x_1)$ drives the internal forces.

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Constructing the Blueprint: Step 3 (Configuration Form)

Packaging raw equations into the standard second-order matrix: $M(d^2q/dt^2) + C(dq/dt) + Kq = F$ in Inter

Mass/Inertia **Dampers/Dissipation** **Springs/Stiffness**

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} (x_1)'' \\ (x_2)'' \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{bmatrix} (x_1)' \\ (x_2)' \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

Physical coupling results in symmetric off-diagonal terms.

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Constructing the Blueprint: Step 4 (State-Space Architecture)

Converting n-degrees of freedom into 2n first-order equations.

Define the State Variables (**z**)

- z_1 = x_1 (Position 1)
- z_2 = (x_1)’ (Velocity 1)
- z_3 = x_2 (Position 2)
- z_4 = (x_2)’ (Velocity 2)

The State Equation Form: **dz/dt = Az + Bu**

$$\begin{bmatrix} z_1' \\ z_2' \\ z_3' \\ z_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k_1+k_2)}{m_1} & -\frac{(c_1+c_2)}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} [f(t)]$$

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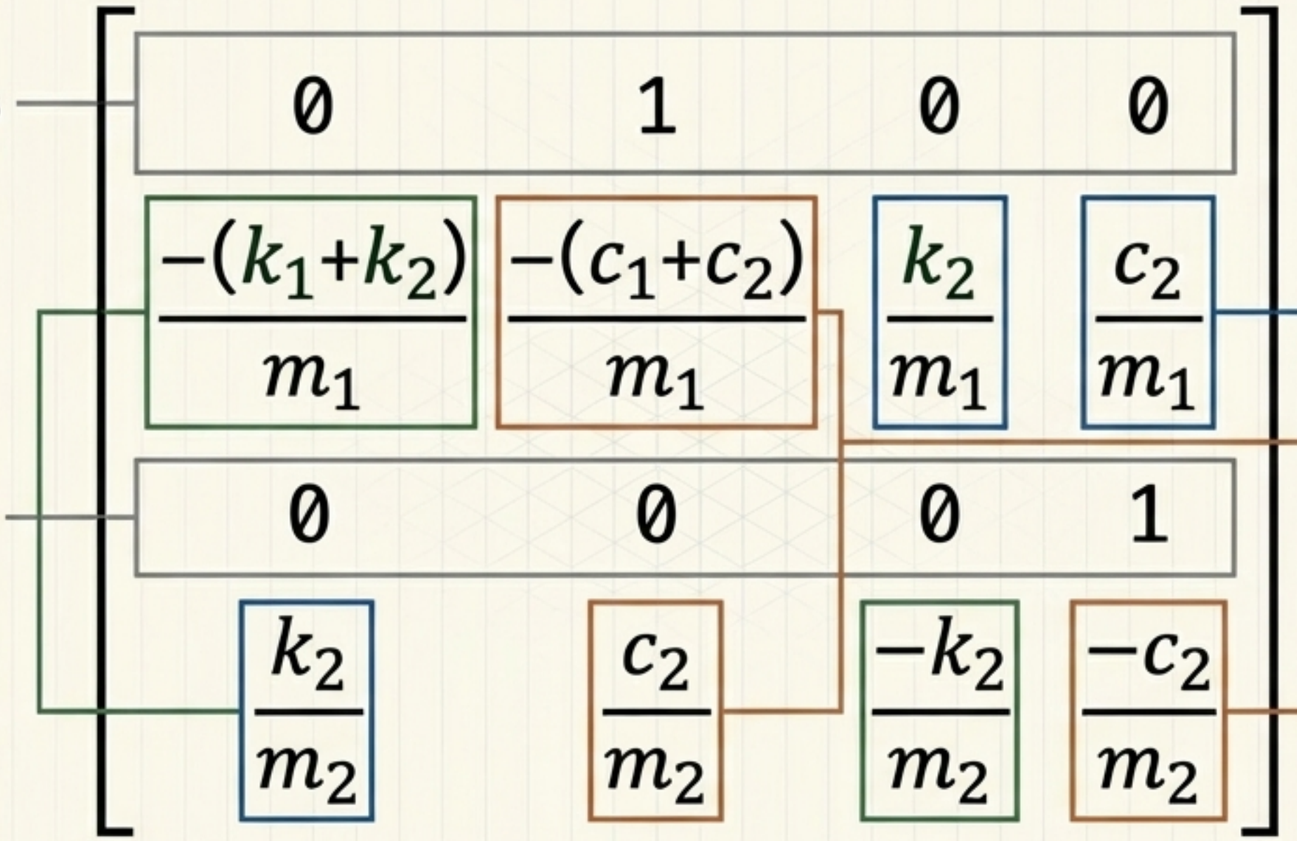
Anatomy of the System Matrix (A)

The Identity Blocks

Kinematics: Simply states that the derivative of position is velocity (z_1' = z_2).

Stiffness Map

Determines how positions generate acceleration.



Inertial Scaling

Every force is divided by mass, fulfilling **a = F/m**.

Damping Map

Determines how velocities generate acceleration.

Core Takeaway: A matrix is not just math; it is a perfectly organized, highly structured container for physical geometry.

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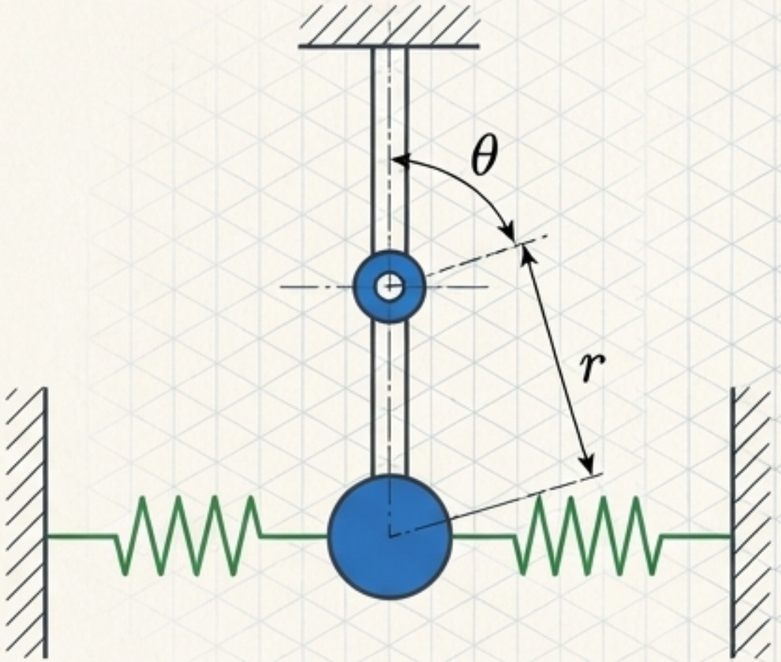
From Matrix to Insight: System Response

The Rotational System



By applying the matrix translation to this rotational system, we derive its core equation of motion:

$$(J_G + mr^2)\theta'' + 4kr^2\theta = 0$$

The Output Insight (Natural Frequency)

$$\omega_n = \sqrt{\frac{4kr^2}{J_G + mr^2}}$$

The model yields the natural frequency—the exact 'heartbeat' of the system. We have successfully translated a complex physical assembly into a single, predictable parameter ready for control design.

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