The Languages of Control Engineering
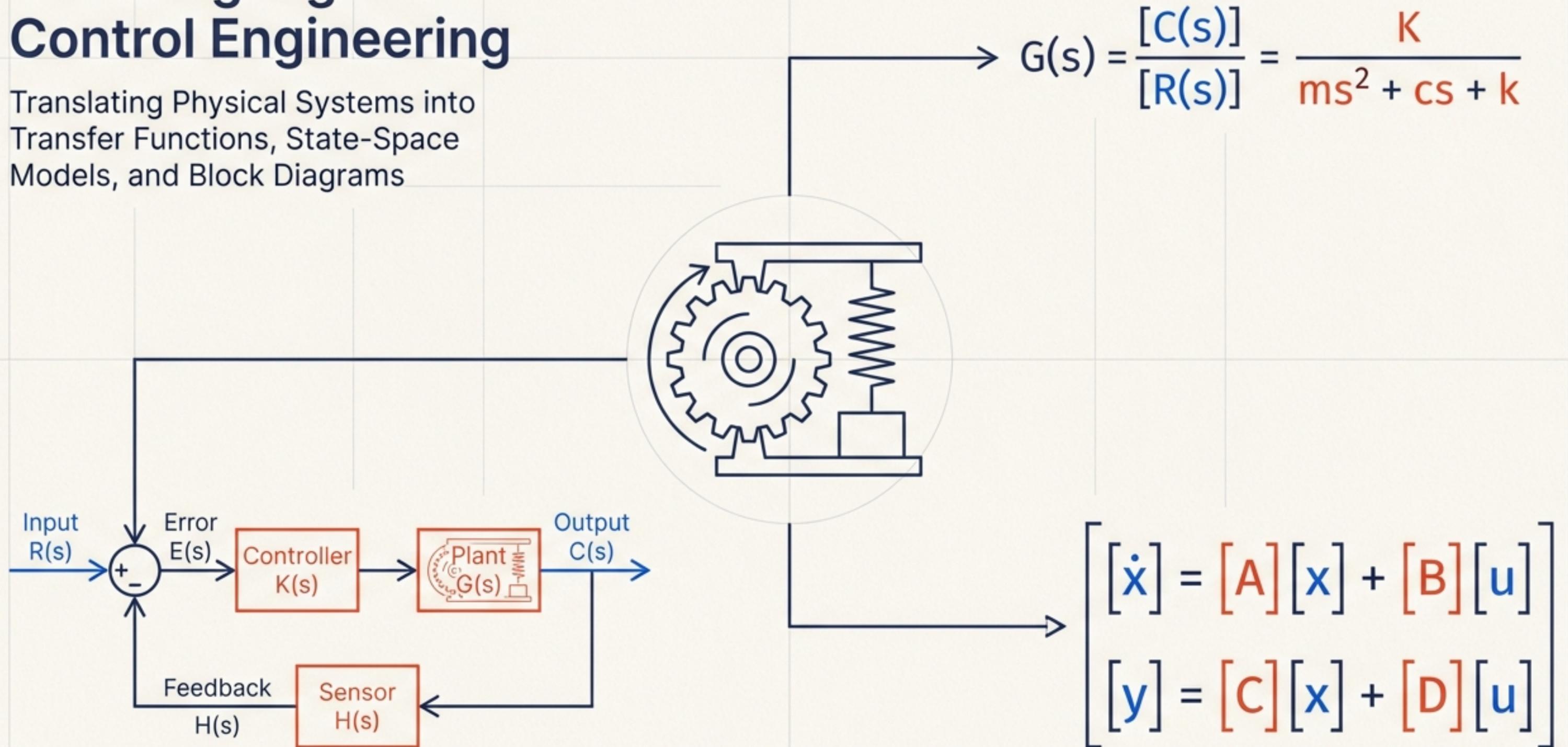
Translating Physical Systems into Transfer Functions, State-Space Models, and Block Diagrams

$$G(s) = \frac{[C(s)]}{[R(s)]} = \frac{K}{ms^2 + cs + k}$$

Input R(s) → Error E(s) → Controller K(s) → Plant G(s) → Output C(s)

Feedback H(s) → Sensor H(s)

$$\begin{bmatrix} \dot{x} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix}\begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} B \end{bmatrix}\begin{bmatrix} u \end{bmatrix}$$
$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} C \end{bmatrix}\begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} D \end{bmatrix}\begin{bmatrix} u \end{bmatrix}$$

# Two Languages, One System

Transfer Function (Classical)

- **Domain:** Frequency / Laplace (s)

- **Capacity:** Single-Input, Single-Output (SISO)

- **Focus:** The Black Box (Input-to-Output ratio only)

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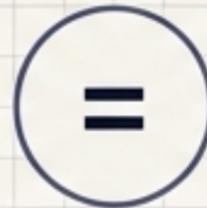
State-Space (Modern)

- **Domain:** Time / Matrix (t)

- **Capacity:** Multiple-Input, Multiple-Output (MIMO)

- **Focus:** Internal Dynamics (Current state predicts next state)

Transfer Functions: The Vending Machine

R(s)

G(s)

Y(s)
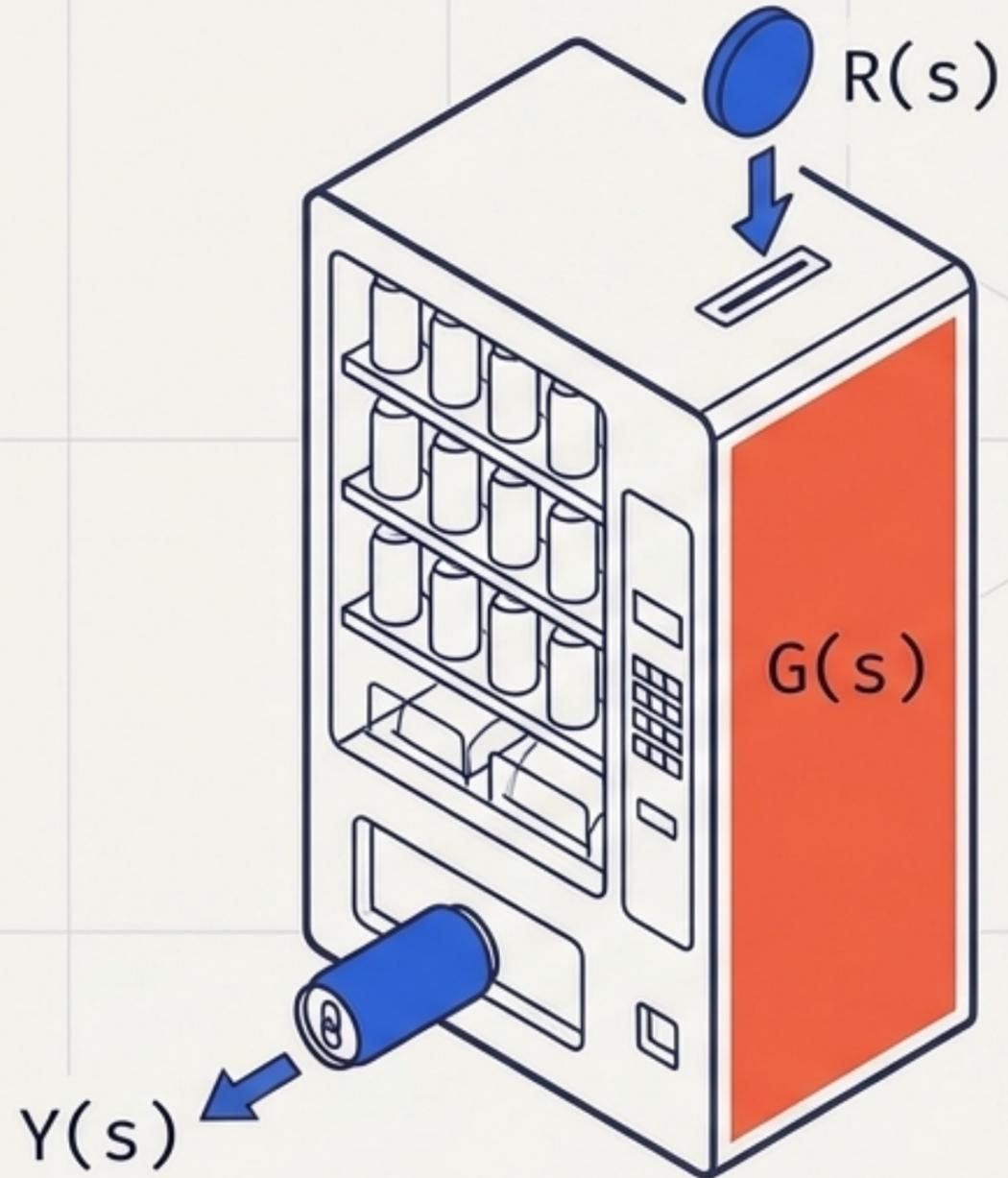
Equation Anatomy

You know the input (coin) and the output (product).

$$G(s) = \frac{Y(s)}{R(s)}$$

You don't need to know the internal gears to know what happens when you press the button.

Core Rule: Assumes all initial conditions are zero.

The Transfer Function Pipeline

Station 1: Differential Equation (Time Domain)

$$2\ddot{x}(t) + 12\dot{x}(t) + 18x(t) = 6f(t)$$

Station 2: Laplace Transform (Frequency Domain)

Action: Assume zero initial conditions.

$$(2s^2 + 12s + 18)X(s) = 6F(s)$$

Station 3: Algebraic Ratio (The Transfer Function)

Action: Isolate Output / Input.

$$G(s) = \frac{X(s)}{F(s)} = \frac{3}{s^2 + 6s + 9}$$

The Characteristic Equation & Stability

$$\frac{3}{\boxed{s^2 + 6s + 9}}$$

The Anatomy

$$s^2 + 6s + 9 = 0$$

Factored Form

$$(s + 3)^2 = 0$$

Visual Math Proof

Roots: $s_1 = s_2 = -3$

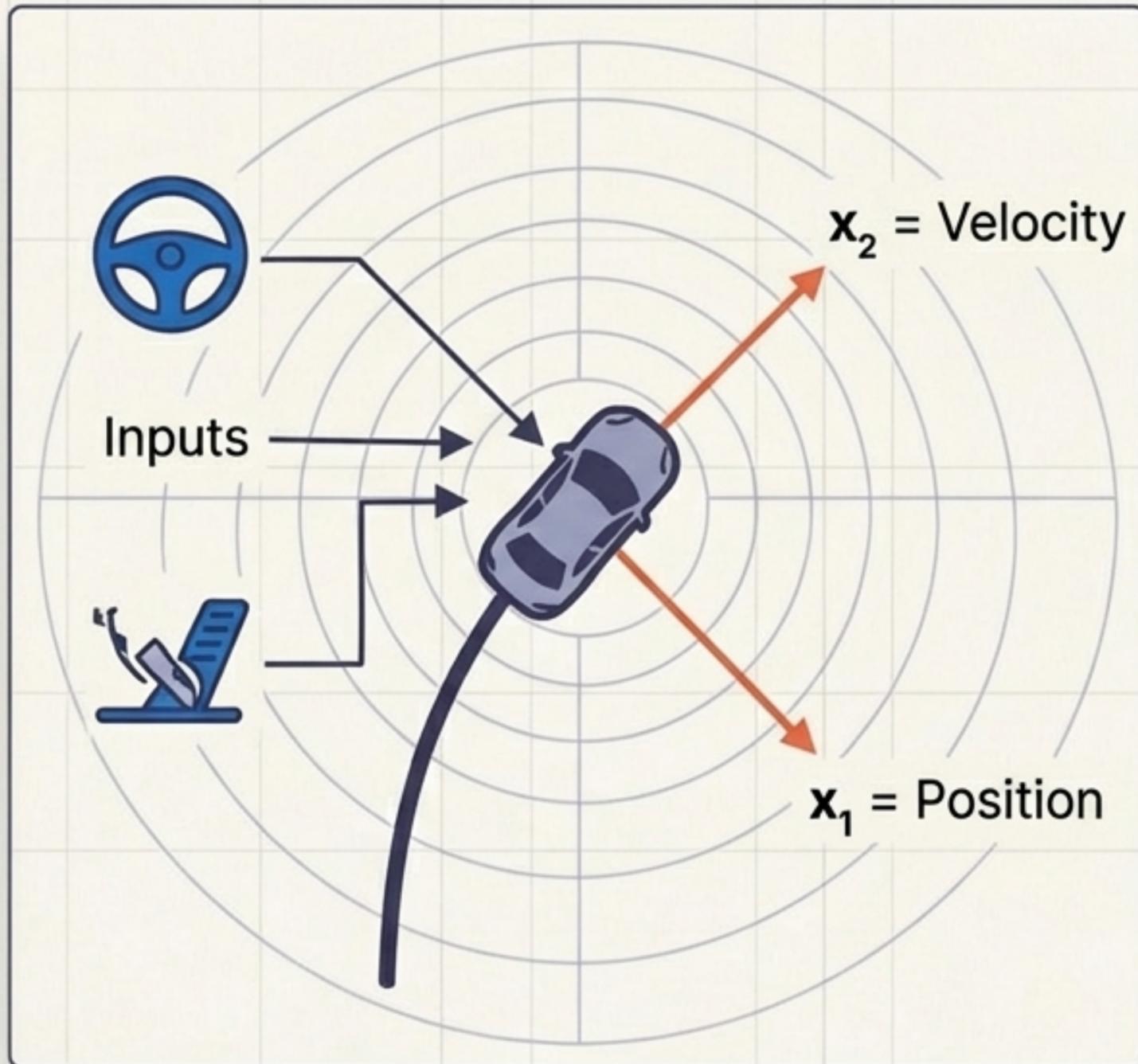Because all roots (poles) have negative real parts, the system is **STABLE**.

Compared to standard form $s^2 + 2\zeta\omega_n s + \omega_n^2$:

Natural frequency $\omega_n = 3$ rad/s

Damping ratio $\zeta = 1$ **(Critically damped)**

State-Space: The GPS Tracker

x_2 = Velocity

Inputs

x_1 = Position

Concept

State variables are your exact current position and velocity. Knowing this "state vector" plus your current input (steering) lets you predict exactly where you will be next.

Equation Anatomy

$$\dot{x} = [A]x + [B]u$$
(State Equation)

$$y = [C]x + [D]u$$
(Output Equation)

NotebookLM

Unpacking the Matrices

[A] The State Matrix
with Helvetica Now Text

Defines internal system dynamics and stability. Contains the characteristic equation roots/eigenvalues.

[B] The Input Matrix
with Helvetica Now Text

Defines how external inputs (actuators) connect to and affect the internal state variables. Determines controllability.

[C] The Output Matrix

Maps internal state variables to the actual outputs we can measure (sensors).

[D] Direct Transmission Matrix

Represents inputs that bypass the system dynamics to directly affect the output (often equals 0).

Translating TF -> State-Space

Start
$$G(s) = \frac{4}{s^2 + 3s + 2}$$

Step 1: Convert to ODE

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 4u(t)$$

Step 2: Define State Variables

$$z_1 = y(t) \qquad z_2 = \dot{y}(t)$$

Step 3: Build Matrices

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u$$

Translating State-Space → TF

The Master Formula

$$G(s) = [C](sI - [A])^{-1}[B] + [D]$$

Step 1: The Transition Matrix (sI - A)

$$\begin{bmatrix} s & -1 \\ 5 & s+6 \end{bmatrix}$$

Step 2: The Inverse

$$\frac{1}{s^2 + 6s + 5} \begin{bmatrix} s+6 & 1 \\ -5 & s \end{bmatrix}$$

Step 3: Multiply by C and B
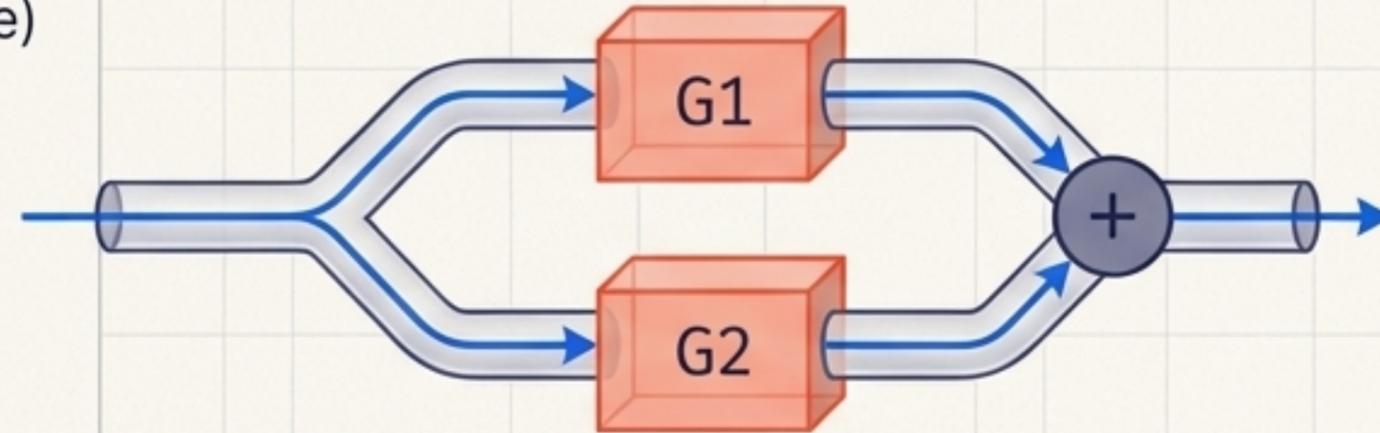
$$G(s) = \frac{1}{s^2 + 6s + 5}$$

NotebookLM

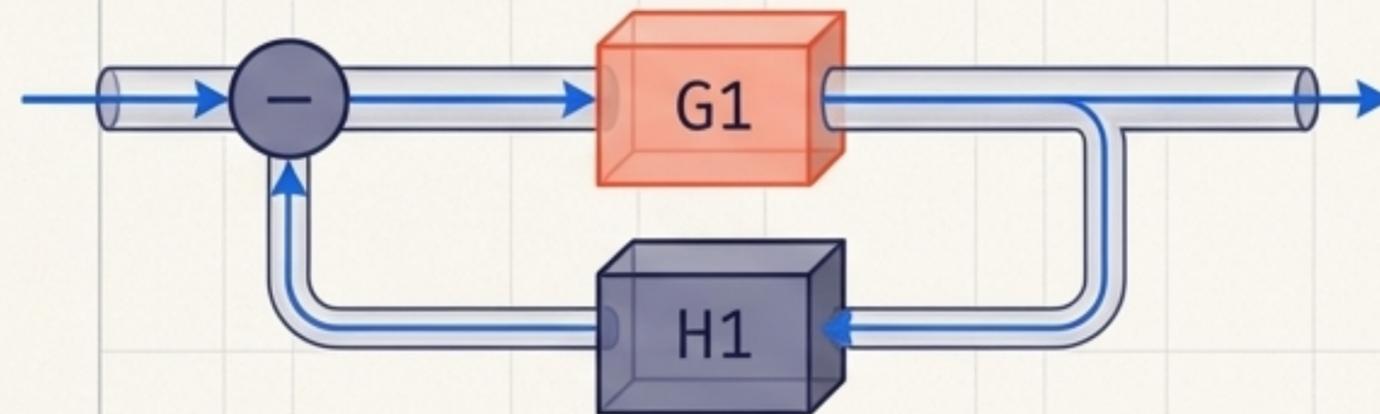# Block Diagram Algebra: The Plumbing System

1. Series (Sequential Pipes)

$$G_{equivalent} = G1 \times G2$$

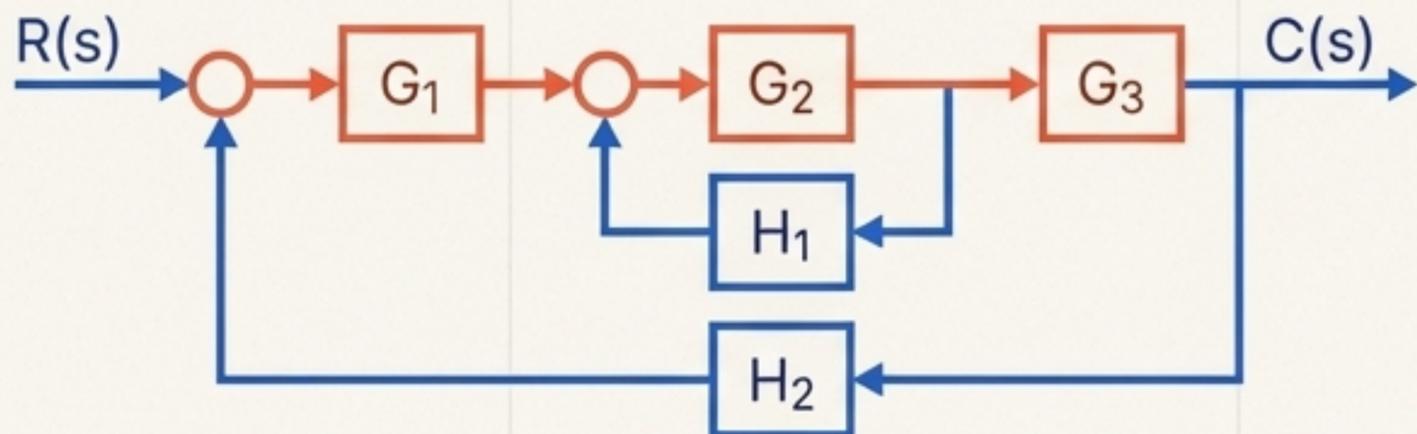**2. Parallel** (Split and Recombine)

$$G_{equivalent} = G1 + G2$$

3. Feedback (The Thermostat)
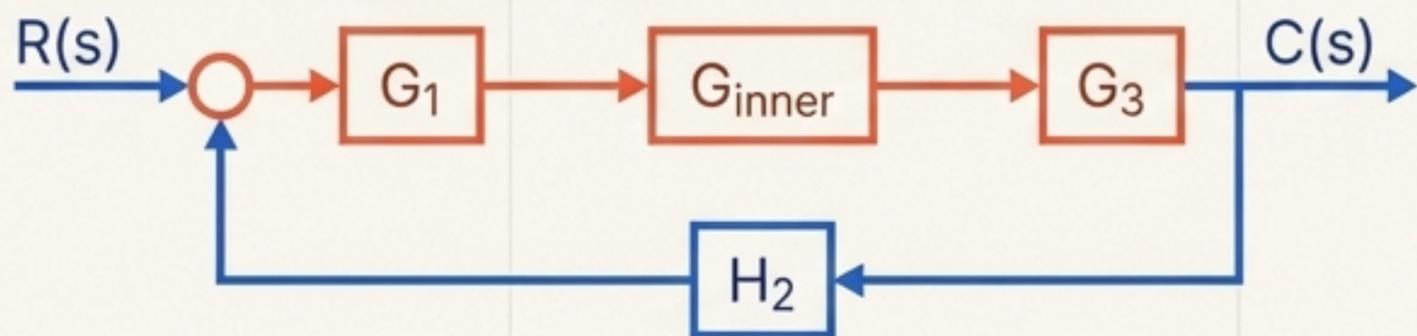

$$G_{equivalent} = \frac{G1}{1 \pm G1\, H1}$$
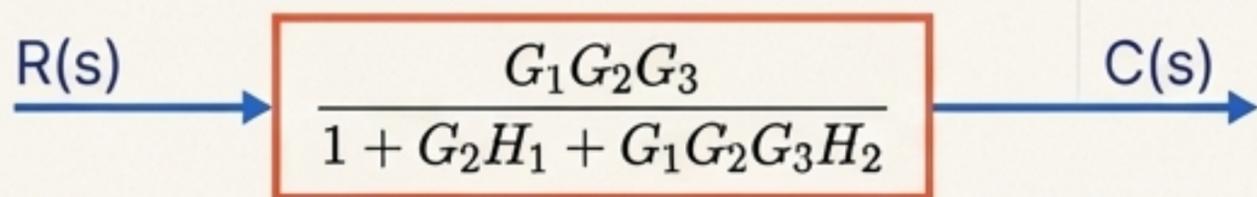
Reducing Complex Loops

Phase 1: Isolate the Inner Loop

Phase 2: Cascade with Forward Blocks

Phase 3: Close the Outer Loop

Collapse inner loop G_2 and H_1:

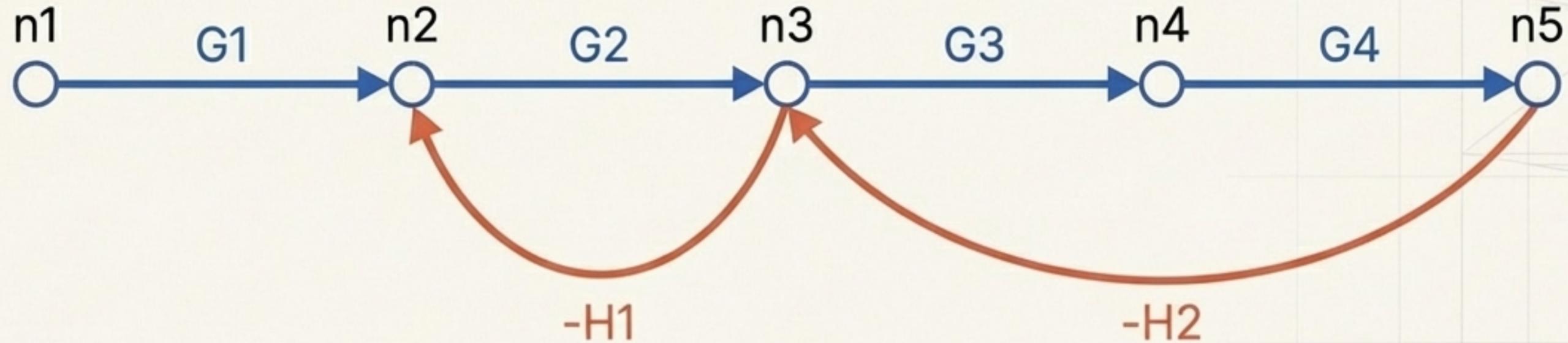$$G_{\text{inner}} = \frac{G_2}{1 + G_2 H_1}$$

Multiply:

$$G_{\text{forward}} = G_1 \times G_{\text{inner}} \times G_3$$

Final Transfer Function:

$$\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3}{1 + G_2 H_1 + G_1 G_2 G_3 H_2}$$

NotebookLM

Signal Flow Graphs

Concept: When block diagrams become too complex, we map them as nodes (variables) and branches (gains).

Forward Paths (P_k)

Direct routes from input to output without crossing the same node twice.

Example: $P_1 = G_1 G_2 G_3 G_4$

Individual Loops (L_m)

Closed paths that start and end at the same node.

Example:
$$L_1 = -G_2 H_1, \quad L_2 = -G_3 G_4 H_2$$

Non-touching Loops

Loops that share no common nodes.

Mason's Gain Formula

$$T = \frac{\sum P_k \Delta_k}{\Delta}$$

Δ (The Determinant)

1 - (sum of all loops) + (sum of non-touching loop pairs) - ...

Example:
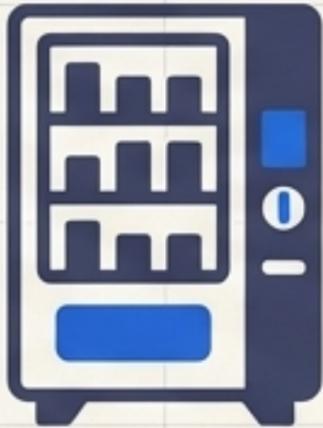$\Delta = 1 - (L_1 + L_2) + (L_1 \times L_2)$

Δ_k (The Co-factor)

The determinant of the graph with loops touching the k-th forward path removed. (If all loops touch the path, $\Delta_k = 1$).

Final Output

$$\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4}{1 + G_2 H_1 + G_3 G_4 H_2 + G_2 G_3 G_4 H_1 H_2}$$
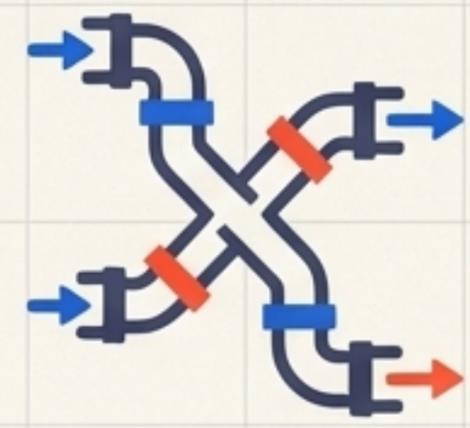
The Engineer's Toolkit

Transfer Functions (The Black Box)

Use for SISO systems, frequency domain analysis, and quick stability checks (Characteristic Equation).

State-Space (The Tracker)

Use for MIMO systems, time-domain control, and manipulating internal system states via Matrices A, B, C, D.

System Manipulation (The Plumbing)

Use Block Diagram Algebra or Mason's Gain Formula to connect, collapse, and calculate the final system output.