Approximate Methods for the Fundamental Frequency

Influence coefficients · Dunkerley · Rayleigh

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MEC521 Vibrations · Handbook Chapter 5 · §5.1–5.3

Semester 20264

How to use this chapter

Three documents, three different jobs. Use the right one.

These slides	Where every formula *comes from*, and what it means. Follow the derivations here.
The handbook	*Nota padat* — the compact reference. Look up a result, a tag, a worked example. Not a first read.
The tutorial	Where you find out whether you actually understood it.

This topic is the standing Question 5

Every conventional MEC521 final paper on file — 2016 through 2024 — examines Dunkerley, Rayleigh and influence coefficients as its last question, for 10 to 20 marks.

Where Chapter 5 is going

Chapter 3 found natural frequencies *exactly*, from a determinant. That works for two degrees of freedom and becomes impractical fast.

1. **Describe the system by its flexibility**, not its stiffness — influence coefficients. §5.1
2. **Dunkerley**: a quick estimate that is always a little *low*. §5.2
3. **Rayleigh**: an estimate that is always a little *high*. §5.3
4. **Use both** and you have bracketed the true answer. §5.2–5.3
5. **Or compute it**, in twenty lines of code. §5.4

Where we are

Chapter	Question	Method
1–2	one mass, exactly	$\omega_n = \sqrt{k/m}$
3	two masses, exactly	$\left\lvert [k] - \omega^2 [m] \right\rvert = 0$
5	**many masses, approximately**	**Dunkerley, Rayleigh**

Why approximately is often enough

In design you rarely need the fundamental frequency to four figures. You need to know whether it is near the running speed — and you need to know that before the detailed model exists.

AIMS topic 5.0, taught in weeks 10–11 inside the Chapter 3 block.

Why bother, when a computer can do it exactly?

What these methods give you

- An answer on the back of an envelope, in minutes.
- A *bound* — you know which side of the truth you are on.
- A sanity check on the finite-element result someone hands you.

Why that still matters

- An FE model that is out by a factor of two looks entirely plausible.
- Dunkerley takes five minutes and catches it.
- Nobody checks a number they have no independent estimate for.

The value is not the number. It is knowing whether to believe the other number.

Why only the lowest frequency?

A pump on a factory floor

A shaft carries a pulley and an impeller between two bearings. The motor starts it from rest and accelerates it to running speed:

- a 4-pole motor on the 50 Hz supply: $1450\,\text{rev/min} = 151.8\,\text{rad}\,\text{s}^{-1}$;
- a 2-pole motor: $2900\,\text{rev/min} = 303.7\,\text{rad}\,\text{s}^{-1}$.

What happens on the way up

Every natural frequency is a speed at which the shaft resonates and *whirls*. Climbing from zero, the **first** one is the first you meet — and the higher ones are usually far above any running speed.

You already know the trend

Stand on a diving board and it bounces more slowly; load a bookshelf plank and it sags and sways more slowly. **Adding mass lowers the fundamental.** With many masses, which is it, and is it near the running speed? That one number is what this chapter estimates.

The problem with starting from $[k]$

For a chain of springs, $[k]$ is easy — you saw that in Chapter 3. For a **shaft carrying discs**, it is not. What is the "stiffness" between disc 1 and disc 2 on a continuous beam? There is no spring to point at.

So turn the question round

Instead of asking *what force produces a unit deflection*, ask *what deflection is produced by a unit force*. That is a question beam theory answers directly.

The result is the **flexibility matrix** $[a]$ — and it is the inverse of $[k]$.

Flexibility influence coefficients

Definition

a_{ij} is the deflection **at station** i caused by a **unit load at station** j, with no other loads applied.

By superposition, any set of loads gives

$$\{x\} = [a]\{F\}, \qquad x_i = \sum_j a_{ij} F_j \qquad \text{F5.1.1}$$

$[k]$	apply unit *displacements*, read off forces
$[a]$	apply unit *forces*, read off displacements

Two descriptions of the same system. Use whichever is easier to write.

Push here, measure there

An influence coefficient is an experiment you could do on the bench:

1. hang a known weight at station j;
2. read a dial gauge at station i;
3. divide the reading by the weight. That is a_{ij}, in $\mathrm{m\,N^{-1}}$.

Why "per newton"

On this chapter's shaft (0.9 m span, 50 mm diameter), 1 N at the 40 kg disc moves the 60 kg disc by only 0.16 µm. Real loads are thousands of newtons — so we tabulate the response to one newton and multiply.

$$i = \text{where you } \textit{measure}, \quad j = \text{where you } \textit{push}.$$

Maxwell's reciprocal theorem

$$\boxed{a_{ij} = a_{ji}} \qquad \text{F5.1.2}$$

The deflection at 1 due to a unit load at 2 **equals** the deflection at 2 due to a unit load at 1 — however unsymmetrical the structure looks.

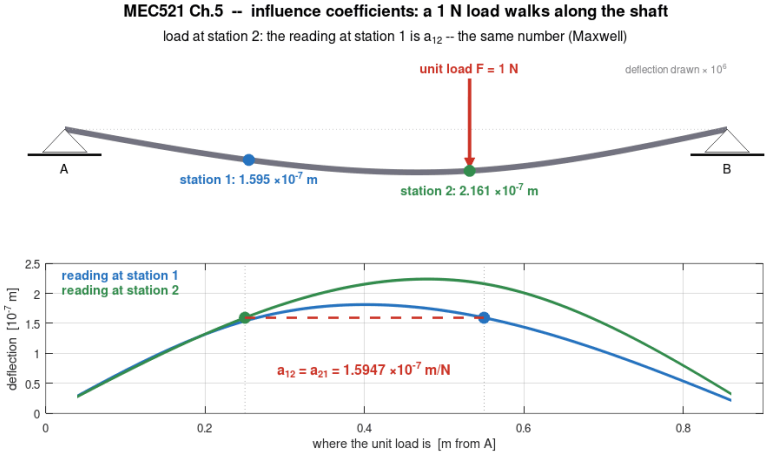
Why it is true

It follows from strain energy: the work done is the same whichever load is applied first. Any linear elastic structure obeys it.

Your strongest free check

Compute a_{12} and a_{21} independently. If they differ, the beam algebra is wrong. This catches more errors in Q5 than anything else.

Watch a 1 N load walk along the shaft



Bottom: what the two dial gauges read as the load moves. Parked at disc 2, the gauge at disc 1 reads exactly what the gauge at disc 2 read with the load at disc 1 — Maxwell, seen. GIF: `ch5-anim/MEC521-ch5-inf.gif`.

The two matrices are inverses

$$[k] = [a]^{-1} \qquad \Longleftrightarrow \qquad [a]\,[k] = [I] \qquad \text{F5.1.3}$$

Which makes sense: apply forces to get displacements, then apply those displacements to get the forces back.

Practical consequence

Build whichever matrix the geometry makes easy, and invert if you need the other. For a shaft, always build $[a]$.

And $[a]\,[k] = [I]$ is a complete, independent check on *both* matrices at once.

Worked example — flexibility of a three-mass chain (1 of 2)

Three masses in a chain to a wall, springs k_1, k_2, k_3. Apply a unit load at one mass at a time.

Unit load at m_1. The load must reach the wall, so only k_1 carries it and stretches $1/k_1$. Springs k_2, k_3 lie *beyond* the load and carry nothing — so m_2 and m_3 simply ride along:

$$a_{11} = a_{21} = a_{31} = \frac{1}{k_1}$$

Unit load at m_2. Now k_1 and k_2 are both in the load path; k_3 is not:

$$a_{12} = \frac{1}{k_1}, \qquad a_{22} = a_{32} = \frac{1}{k_1} + \frac{1}{k_2}$$

Unit load at m_3. All three carry it:

$$a_{13} = \frac{1}{k_1}, \quad a_{23} = \frac{1}{k_1} + \frac{1}{k_2}, \quad a_{33} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

Worked example — flexibility of a three-mass chain (2 of 2)

$$[a] = \begin{bmatrix} \frac{1}{k_1} & \frac{1}{k_1} & \frac{1}{k_1} \\ \frac{1}{k_1} & \frac{1}{k_1} + \frac{1}{k_2} & \frac{1}{k_1} + \frac{1}{k_2} \\ \frac{1}{k_1} & \frac{1}{k_1} + \frac{1}{k_2} & \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \end{bmatrix}$$

Symmetric ✓— Maxwell holds, as it must.

The check that closes the loop. With $k_1 = 100$, $k_2 = 200$, $k_3 = 300\,\mathrm{N\,m^{-1}}$:

$$\begin{bmatrix} 0.0100 & 0.0100 & 0.0100 \\ 0.0100 & 0.0150 & 0.0150 \\ 0.0100 & 0.0150 & 0.0183 \end{bmatrix} \begin{bmatrix} 300 & -200 & 0 \\ -200 & 500 & -300 \\ 0 & -300 & 300 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Built both ways, independently, and they invert each other.

When the coordinates are not all displacements

A rigid bar on two springs can *translate* and *rotate*. One coordinate is a length, the other an angle.

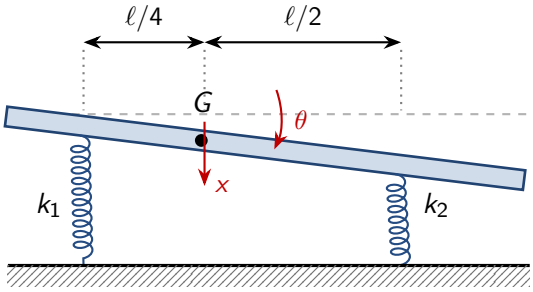
The influence coefficients then have mixed units

a_{11} is $\mathrm{m\,N^{-1}}$; a_{22} is $\mathrm{rad\,N^{-1}\,m}$; the cross terms are $\mathrm{rad\,N^{-1}}$ and $\mathrm{m\,N^{-1}\,m}$.

Maxwell still holds — numerically

$a_{12} = a_{21}$ as *numbers*, because the generalised force conjugate to an angle is a moment. A legacy answer scheme in the archive gets this wrong and produces an asymmetric $[a]$; the symmetry check is what exposes it.

A rigid bar on two springs



For a shaft: the formula you will use

For a simply supported shaft, a load at distance a from one bearing and b from the other, with $a + b = \ell$:

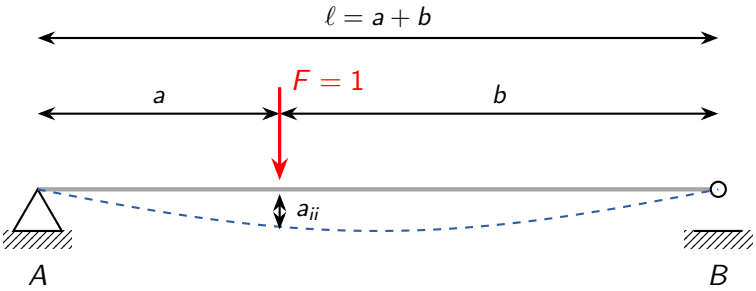
$$\boxed{a_{ii} = \frac{a^2 b^2}{3EI\ell}} \qquad \textbf{F5.1.4}$$

and for the deflection at a *different* station x:

$$a_{ij} = \frac{bx\left(\ell^2 - b^2 - x^2\right)}{6EI\ell} \qquad \textbf{F5.1.5}$$

In **F5.1.5**, b belongs to the *loaded* station and x locates the station where you want the deflection, measured from the *same* bearing. Getting that convention wrong is the classic Q5 error.

The shaft influence coefficient



The eigenvalue problem, in flexibility form

Starting from $\{x\} = [a]\{F\}$ with $\{F\} = -[m]\{\ddot{x}\}$ and a harmonic trial:

$$[a][m]\{X\} = \frac{1}{\omega^2}\{X\} \qquad \text{F5.1.6}$$

Note which way round the eigenvalue sits

The eigenvalues of $[a][m]$ are $1/\omega^2$, so the **largest** eigenvalue corresponds to the **lowest** — the fundamental — frequency.

That inversion is exactly why the flexibility formulation is the natural home for fundamental-frequency estimates.

Dunkerley's formula

$$\frac{1}{\omega_1^2} \simeq \sum_i m_i a_{ii} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} + \cdots \qquad \text{F5.2.1}$$

Each ω_{ii} is the natural frequency the system would have with **only mass i present**.

Compute them one at a time and add the reciprocal squares.

Why it is so easy to use

It needs only the **diagonal** influence coefficients. No cross terms, no matrix, no determinant.

Where Dunkerley's formula comes from

Expand the characteristic equation of $[a]\,[m]$ in terms of $\lambda = 1/\omega^2$. The sum of the eigenvalues is the trace:

$$\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \cdots = \operatorname{tr}\left([a]\,[m]\right) = \sum_i m_i a_{ii}$$

Now discard everything but the first mode. Since $\omega_1 \ll \omega_2$, the higher terms $1/\omega_2^2, \ldots$ are small:

$$\frac{1}{\omega_1^2} \simeq \sum_i m_i a_{ii}$$

What you threw away was positive

So the left-hand side is over-estimated, $1/\omega_1^2$ comes out too big, and ω_1 comes out **too small**.

Why Dunkerley is low, with numbers

For the chapter's two-disc shaft the trace identity is *exact*:

$$\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} = m_1 a_{11} + m_2 a_{22} = 1.7884 \times 10^{-5}\,\mathrm{s}^2$$

The second mode is at $\omega_2 = 943.1\,\mathrm{rad\,s^{-1}}$, so its term is small — but not zero: it is 6.3 % of the total.

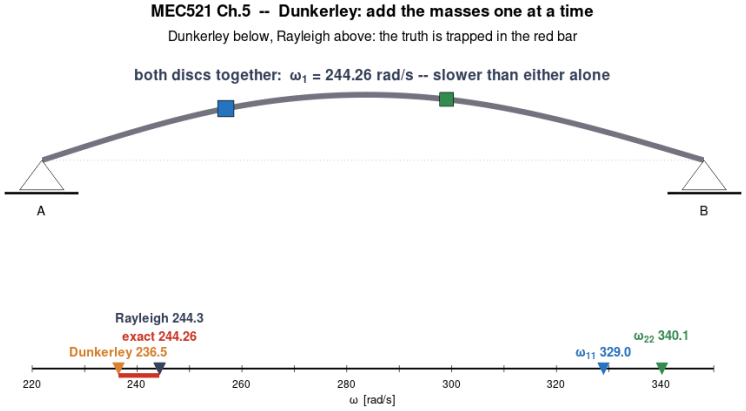
Dunkerley hands *all* of that total to the first mode. Its $1/\omega_1^2$ is therefore too big, and its ω_1 too small:

$$236.5\,\mathrm{rad\,s^{-1}}\ \text{(Dunkerley)} \quad < \quad 244.26\,\mathrm{rad\,s^{-1}}\ \text{(exact)}$$

In plain words

Each disc alone gives the shaft only one load to carry, so it is stiff and fast (328.98 and $340.12\,\mathrm{rad\,s^{-1}}$). Together they are slower than either — and Dunkerley, by ignoring the higher modes, overshoots the slowing a little.

Watch the masses join



One disc at a time, then both. The number line collects ω_{11}, ω_{22}, Dunkerley, the exact answer and Rayleigh: the truth ends up trapped between the two estimates. GIF: `ch5-anim/MEC521-ch5-dun.gif`.

Dunkerley is always LOW — and that is useful

$$\omega_{\text{Dunkerley}} \leq \omega_1 \qquad \text{F5.2.2}$$

It is a **lower bound**, not a guess. The true fundamental frequency is never below it.

Which makes it a safe check

If Dunkerley says $236\,\text{rad}\,\text{s}^{-1}$ and your running speed is $300\,\text{rad}\,\text{s}^{-1}$, you are *not* safe — the true value is somewhere above 236 and could be anywhere. If Dunkerley says $236\,\text{rad}\,\text{s}^{-1}$ and you run at $100\,\text{rad}\,\text{s}^{-1}$, you are safe with certainty.

Worked example — Dunkerley (1 of 3)

A steel shaft, $d = 50\,\text{mm}$, in bearings 900 mm apart. A 60 kg disc at 250 mm from A, a 40 kg disc at 550 mm from A. $E = 207\,\text{GPa}$; shaft mass neglected.

Step 1 — second moment of area

$$I = \frac{\pi d^4}{64} = \frac{\pi (0.050)^4}{64} = 3.0680 \times 10^{-7}\,\text{m}^4$$

Step 2 — flexural rigidity

$$EI = 207 \times 10^9 \times 3.0680 \times 10^{-7} = 63\,507\,\text{N m}^2$$

$I = \pi d^4/64$, not $\pi d^4/32$

Bending, not torsion. The factor of two from Chapter 1 returns here.

Step 3 — the diagonal influence coefficients. For each disc, a and b are its distances to the two bearings, $a + b = 0.9\,\text{m}$.

Disc 1: $a = 0.25$, $b = 0.65$:

$$a_{11} = \frac{(0.25)^2(0.65)^2}{3(63507)(0.9)} = 1.5400 \times 10^{-7}\,\text{m N}^{-1}$$

Disc 2: $a = 0.55$, $b = 0.35$:

$$a_{22} = \frac{(0.55)^2(0.35)^2}{3(63507)(0.9)} = 2.1611 \times 10^{-7}\,\text{m N}^{-1}$$

Step 4 — the single-mass frequencies

$$\omega_{11} = \frac{1}{\sqrt{m_1 a_{11}}} = \frac{1}{\sqrt{60(1.5400 \times 10^{-7})}} = 328.98\,\text{rad s}^{-1}$$

$$\omega_{22} = \frac{1}{\sqrt{40(2.1611 \times 10^{-7})}} = 340.12\,\text{rad s}^{-1}$$

Worked example — Dunkerley (3 of 3)

Step 5 — add the reciprocal squares

$$\frac{1}{\omega_1^2} \simeq \frac{1}{328.98^2} + \frac{1}{340.12^2} = 9.2400 \times 10^{-6} + 8.6444 \times 10^{-6} = 1.7884 \times 10^{-5}\ \mathrm{s}^2$$

$$\boxed{\omega_1 \simeq 236.5\,\mathrm{rad\,s^{-1}} \qquad f_1 \simeq 37.6\,\mathrm{Hz}}$$

Step 6 — how good is it?

The exact answer, from the full eigenvalue problem, is $\omega_1 = 244.3\,\mathrm{rad\,s^{-1}}$.

Dunkerley is 3.2 % **low** — as it must be.

Note that *both* single-mass frequencies are far above the answer. Adding masses always lowers the fundamental.

Rayleigh's method: the principle

At a natural frequency the system exchanges all its energy between kinetic and potential. Equate the two maxima:

$$T_{\max} = V_{\max} \quad \Longrightarrow \quad \omega^2 \simeq \frac{\{X\}^{\mathrm{T}}\,[k]\,\{X\}}{\{X\}^{\mathrm{T}}\,[m]\,\{X\}} \qquad \text{F5.3.4}$$

Assume a *shape* $\{w\}$ for the first mode. Then

$$\omega_1^2 \simeq \frac{g\sum_i m_i w_i}{\sum_i m_i w_i^2} \qquad \text{F5.3.1}$$

The Rayleigh quotient

Feed in any shape and it returns a frequency. Feed in the *true* mode shape and it returns the exact answer.

Where the Rayleigh quotient comes from

Assume every disc moves in step, in a fixed shape w_i: $x_i(t) = w_i \cos \omega t$.

Kinetic energy is largest as the shaft flies through straight ($\cos \omega t = 0$, speed ωw_i):

$$T_{\max} = \tfrac{1}{2}\omega^2 \sum_i m_i w_i^2$$

Potential energy is largest at the extreme, where the shaft is bent into the shape w_i. If that shape is the static deflection under the weights $m_i g$, the stored energy is the work those weights did:

$$V_{\max} = \tfrac{1}{2} \sum_i (m_i g)\, w_i$$

Equate them — energy only swaps between the two:

$$\omega^2 = \frac{g \sum_i m_i w_i}{\sum_i m_i w_i^2} \qquad \text{F5.3.1}$$

Why Rayleigh is always HIGH

$$\omega_{\text{Rayleigh}} \geq \omega_1 \qquad \text{F5.3.2}$$

Any trial shape that is not the true mode is a *mixture* of the true modes — and every other mode has a higher frequency than the first. So the quotient returns a weighted average that is pulled **upward** by the contamination.

And the error is second-order

An error of 10 % in the assumed shape produces an error of well under 1 % in the frequency. That is why a crude guess works so well.

The best cheap guess: the static deflection curve

The shape the structure takes under its *own weight* is remarkably close to its first mode — both are dominated by the same flexibility.

$$w_i = g \sum_j m_j a_{ij} \qquad \text{F5.3.3}$$

This is where the cross terms finally matter

Dunkerley needed only a_{ii}. Rayleigh needs the full row — including a_{ij} — because the deflection at station i depends on *every* mass.

Which is the price of the better answer: one more influence coefficient to compute.

Worked example — Rayleigh (1 of 2)

The same shaft: $\ell = 0.9\,\text{m}$, $EI = 63\,507\,\text{N m}^2$, $m_1 = 60\,\text{kg}$ at $0.25\,\text{m}$, $m_2 = 40\,\text{kg}$ at $0.55\,\text{m}$.

Step 1 — the cross-coefficient Dunkerley discarded. With $x = 0.25$ (where we want the deflection), and the load at disc 2 so $b = 0.35$:

$$a_{12} = a_{21} = \frac{(0.35)(0.25)\left[(0.9)^2 - (0.35)^2 - (0.25)^2\right]}{6(63507)(0.9)} = 1.5947 \times 10^{-7}\,\text{m N}^{-1}$$

Step 2 — static deflections under self-weight

$$w_1 = 9.81\left[60(1.5400 \times 10^{-7}) + 40(1.5947 \times 10^{-7})\right] = 1.5322 \times 10^{-4}\,\text{m}$$

$$w_2 = 9.81\left[60(1.5947 \times 10^{-7}) + 40(2.1611 \times 10^{-7})\right] = 1.7867 \times 10^{-4}\,\text{m}$$

Step 3 — feed the shape into the quotient

$$\omega_1 \simeq \left[\frac{9.81\left[60(1.5322\times10^{-4})+40(1.7867\times10^{-4})\right]}{60(1.5322\times10^{-4})^2+40(1.7867\times10^{-4})^2}\right]^{1/2} = \left[\frac{0.16029}{2.6854\times10^{-6}}\right]^{1/2}$$

$$\boxed{\omega_1 \simeq 244.3\,\mathrm{rad\,s^{-1}} \qquad f_1 \simeq 38.9\,\mathrm{Hz}}$$

The exact answer is $244.26\,\mathrm{rad\,s^{-1}}$. Rayleigh is high by **0.02 %**.

A guessed shape, two influence coefficients, and an answer good to four figures.

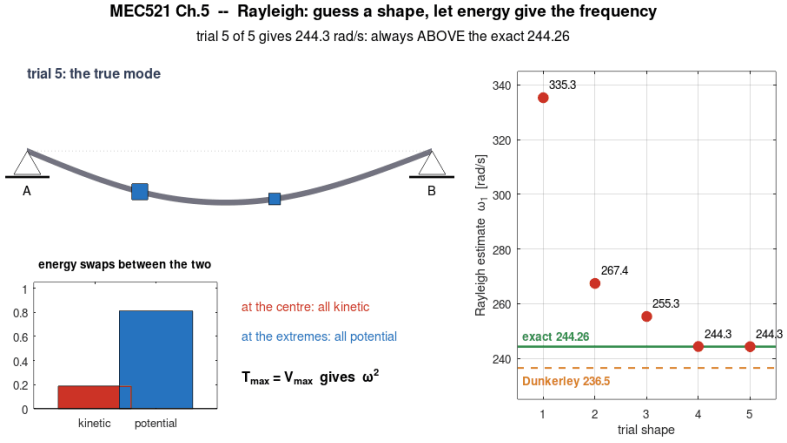
How much does the guess actually matter?

The same shaft, with different trial shapes:

Trial shape	ω_1 (rad s^{-1})	**Error**
static deflection	244.3	+0.02 %
equal displacement $\{1, 1\}$	255.3	+4.5 %
crude linear $\{1, 1.5\}$	267.4	+9.5 %
badly wrong $\{1, 2\}$	335.3	+37 %
the true mode	244.26	exact

Every one is **high**, as the theory guarantees — and the static deflection curve is worth the extra coefficient.

Watch the guesses close in



The table above, animated: each trial shape swings, its kinetic and potential energy swap, and the frequency it implies is plotted. Every point is above the exact line. GIF: `ch5-anim/MEC521-ch5-ray.gif`.

The two bounds together

$$\underbrace{236.5\,\mathrm{rad\,s^{-1}}}_{\text{Dunkerley, low}} \;\le\; \omega_1 \;\le\; \underbrace{244.3\,\mathrm{rad\,s^{-1}}}_{\text{Rayleigh, high}}$$

Two independent hand calculations bracket the true value inside 3 % — and the true answer, $244.26\,\mathrm{rad\,s^{-1}}$, is indeed inside.

This is the real answer to "why bother"

Neither method alone tells you how wrong it is. **Together they do.** No single finite-element run gives you that.

Quote both, and state which is the bound on which side. That is what a full-mark Q5 answer looks like.

Worked example — which motor may drive this shaft?

The same shaft, now driven by one of the two motors from the opening slide.

4-pole, 1450 rev/min**:** $\omega = \dfrac{2\pi(1450)}{60} = 151.8\,\mathrm{rad\,s^{-1}}$. Dunkerley guarantees $\omega_1 \geq 236.5\,\mathrm{rad\,s^{-1}}$, so the shaft runs at most at 64 % of its first critical speed. **Safe, with certainty** — from the lower bound alone.

2-pole, 2900 rev/min**:** $\omega = 303.7\,\mathrm{rad\,s^{-1}}$. Rayleigh guarantees $\omega_1 \leq 244.3\,\mathrm{rad\,s^{-1}}$, so the shaft runs **above** its first critical speed.

Above critical is not automatically wrong — but it has a price

Every start-up and run-down passes *through* resonance. The design must accelerate through it quickly and carry enough damping. Neither bound alone could have told you this; one bound on each side did.

AIMS topic 5.3. Everything in this chapter is a few lines of Octave or Python.

build [a]	one nested loop over the influence-coefficient formula
Dunkerley	`1/sqrt(sum(m.*diag(a)))`
Rayleigh	`w = g*a*m`, then the quotient
exact	`eig(a*diag(m))`, take the largest eigenvalue

The library must run on a bare Octave install

`mec521lib` deliberately avoids every Octave Forge package. If a function needs `signal` or `control`, it is written out longhand.

When the approximate methods stop being worth it

Use Dunkerley/Rayleigh	a handful of discrete masses; you need an answer now; you need a *bound*
Use the eigenvalue solver	more than three or four stations; you need higher modes; you need mode *shapes*
Use finite elements	distributed mass, complex geometry, or the shaft mass is not negligible

And whichever you use, run Dunkerley anyway — it costs five minutes and it is the only independent check you will get.

Three ways to lose marks here

1. $I = \pi d^4/32$ **instead of** $\pi d^4/64$.
   Bending, not torsion. The torsion formula doubles I, so ω comes out $\sqrt{2}$ times too high.

2. **Mixing up x and b in the cross-coefficient.**
   b belongs to the *loaded* station, x to the station being measured, both from the *same* bearing. Check with $a_{12} = a_{21}$.

3. **Reporting a bound without saying which side it is on.**
   Dunkerley is low, Rayleigh is high. Saying so is a marked point.

And one from the units

a_{ij} is in $\mathrm{m\,N^{-1}}$. If yours is not, the EI is wrong.

What the assessment will ask

This semester	not in the Final	Practised in the Chapter 5 tutorial and the Week 11 live lesson.
Past finals	Q5(b), 10–20 marks	Rayleigh vs Dunkerley, influence coefficients, an estimate (to July 2024).

What a full-mark answer looks like

- I and EI computed and their units shown;
- influence coefficients with the a, b, x convention stated;
- Maxwell symmetry checked explicitly;
- both estimates quoted, each labelled as an upper or lower bound;
- a sentence comparing them and saying what that tells you.

What to take away

1. a_{ij} is the deflection at i from a unit load at j, and $a_{ij} = a_{ji}$ always.
2. Dunkerley needs only the diagonal, and comes out **low**.
3. Rayleigh needs the static deflection shape, and comes out **high**.
4. Used together they *bracket* the answer — which no single computation does.

Tutorial: the same shaft with a third disc added.
Predict, before you calculate, whether ω_1 goes up or down.

Handbook § 5.1–5.3. Further reading: Rao, *Mechanical Vibrations*, Ch. 7.

The three you will reach for most: F5.1.4, F5.2.1, F5.3.1.
Everything else is on *the short list* at the front of Appendix A — one page, ordered by what you are looking at.