Two-Degree-of-Freedom Coupled Vibration

Mode shapes · orthogonality · the dynamic vibration absorber

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MEC521 Vibrations · Handbook Chapter 3 · Weeks 8–11

Semester 20264

How to use this chapter

Three documents, three different jobs. Use the right one.

These slides	Where every formula *comes from*, and what it means. Follow the derivations here.
The handbook	*Nota padat* — the compact reference. Look up a result, a tag, a worked example. Not a first read.
The tutorial	Where you find out whether you actually understood it.

This chapter counts in the final

CLO2 is 50 % of the final examination, and two-degree-of-freedom analysis is one of the topics it draws on. Do not skim it.

Where Chapter 3 is going

One number described the whole system in Chapters 1–2. Now it takes two, and almost everything follows from that.

1. **Write two equations instead of one** — and discover they are coupled. § 3.1–3.2
2. **Find TWO natural frequencies** from a determinant. § 3.3
3. **Find the mode shape at each** — how the system moves at that frequency. § 3.3
4. **Use the second mass as a cure**: the dynamic vibration absorber. § 3.5

An n-degree-of-freedom system has n natural frequencies and n mode shapes. Two is where the pattern becomes visible.

One coordinate to two

Chapters 1–2

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

One coordinate. One natural frequency. One mode of vibrating.

Chapter 3

$$[m]\{\ddot{x}\} + [k]\{x\} = \{F\}$$

Two coordinates. **Two** natural frequencies. Two mode shapes.

The algebra is the same shape — the objects changed

Scalars became matrices, and $\omega_n = \sqrt{k/m}$ became an eigenvalue problem. Everything you learned in Chapter 1 still applies, one mode at a time.

Why a second degree of freedom changes everything

What is genuinely new

- **Two** resonances to avoid, not one.
- The system has *preferred shapes* — it does not just move, it moves in a pattern.
- Motion at one mass drives the other. You cannot fix one in isolation.

Why engineers care

- A vehicle body and its wheel: two modes, two very different frequencies.
- A building and a tuned mass damper on the roof.
- Any machine whose foundation is not rigid — which is all of them.

And a second mass, deliberately added, becomes the cure — the absorber.

You have already felt two degrees of freedom

- a car over a **bonggol**: the body *bounces* up and down and *pitches* nose-to-tail — two motions, one body;
- two **swings on one beam** at the playground: push one, and the other starts swinging although nobody touched it;
- a two-storey **shophouse** in strong wind: each floor sways, and each floor pulls on the other.

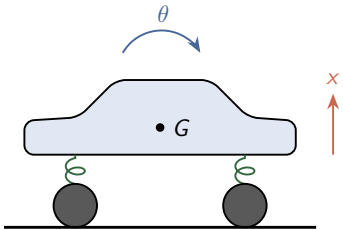
Count the numbers you need

To say where the system is at any instant you need **two** numbers — two positions, or a position and an angle. That count is the number of degrees of freedom.

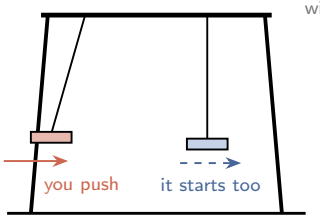
The engineering question

With two numbers moving, how many natural frequencies are there, and what pattern does the system move in at each one? Get this wrong and you design away one resonance while sitting on the other.

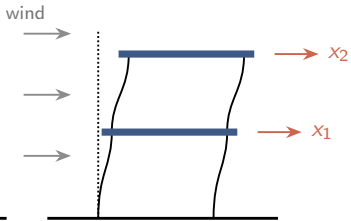
Three things you have already felt



(a) car: bounce and pitch



(b) two swings, one beam



(c) two-storey shophouse

What "coupled" means

Take two masses, each on its own spring to a wall — and a **middle spring** k_2 joining them.

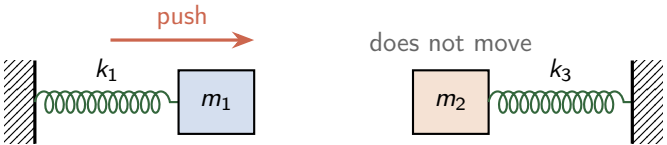
Without the middle spring: push m_1 and only m_1 moves. Two separate Chapter-1 problems.

With it: push m_1, the middle spring stretches, and it pulls on m_2. Mass 2 moves although nobody touched it. That is **coupling**, and it shows up in the equation of mass 1 as a term in the *other* coordinate:

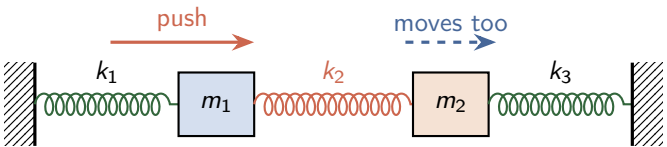
$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 \underbrace{- k_2 x_2}_{\text{the coupling}} = 0$$

Remove k_2 and that term vanishes — the two equations no longer talk to each other.

Take away the middle spring



Without k_2:
two separate
Chapter 1 problems



With k_2:
the term $-k_2x_2$
couples them

What a mode is, before any algebra

Suppose you could set both masses off in a pattern so special that, once released, they move **in step**: both pass through zero together, both reach their peaks together, at one steady frequency.

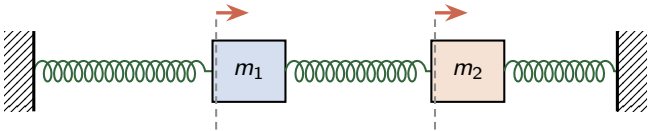
That pattern is a **mode**. Its *shape* is the ratio X_2/X_1; its *frequency* is how fast it repeats.

Two masses, exactly two such patterns

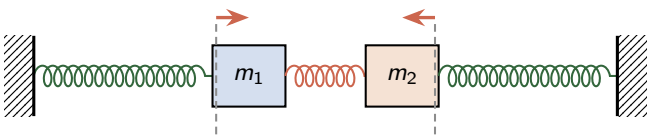
- **both the same way** — the middle spring is barely stretched, so the system feels soft: the **lower** frequency;
- **opposite ways** — the middle spring is squeezed and stretched hard, so it feels stiff: the **higher** frequency.

The rest of this chapter is just how to find those two patterns, and their frequencies, with numbers.

The two patterns, drawn



Both the same way
the middle spring barely changes
feels soft: **lower** frequency



Opposite ways
the middle spring is squeezed hard
feels stiff: **higher** frequency

Dashed lines: rest positions. Half a cycle later, every arrow reverses. Sizes not to scale.

Two ways to be a two-degree-of-freedom system

Two masses

Two bodies, one coordinate each. A chain of masses and springs; a quarter-car with body and wheel.

One body, two motions

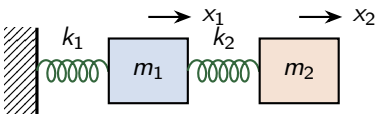
A single rigid body that can both *translate* and *rotate* — a car body bouncing and pitching on two suspensions.

Same mathematics either way

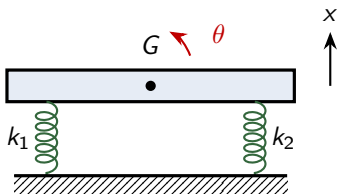
Both give two second-order equations and a 2×2 mass and stiffness matrix. Which physical picture you started from stops mattering the moment you have $[m]$ and $[k]$.

In the second case one coordinate is a length and the other an angle — so the "mass" matrix contains a moment of inertia.

Two masses, or one body moving two ways



(a) two masses, two translations



(b) one body, translation + rotation

Coupling: what the off-diagonal terms mean

Coupling is simply this: **one equation contains the other coordinate**.

$$k_{12} = k_{21} \neq 0 \implies \textbf{static (elastic) coupling} \qquad \text{F3.1.1}$$

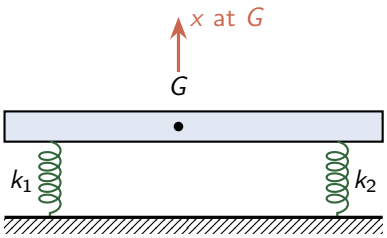
$$m_{12} = m_{21} \neq 0 \implies \textbf{dynamic (inertia) coupling} \qquad \text{F3.1.2}$$

Coupling is a property of the COORDINATES, not of the system

Choose different coordinates and the coupling changes. Choose the right ones — the **principal coordinates** — and both matrices are diagonal and the system falls apart into two independent SDOF problems. F3.1.3

That is exactly what the mode shapes will turn out to be.

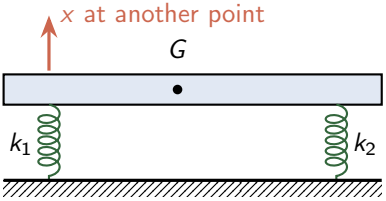
Same body, different coupling



measure x at G

$$[m] = \begin{bmatrix} \times & 0 \\ 0 & \times \end{bmatrix} \quad [k] = \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix}$$

static (elastic) coupling only



measure x somewhere else

$$[m] = \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix} \quad [k] = \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix}$$

dynamic *and* static coupling

Same body, same two natural frequencies. $\times$: non-zero in general.

Deriving the equations: one free body at a time

Step 1. Displace *both* masses in the positive direction, and assume $x_2 > x_1$. Commit to that assumption and keep it.

Step 2. Free body for m_1. The middle spring is stretched by $(x_2 - x_1)$, so it *pulls* m_1 towards m_2:

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1)$$

Step 3. Free body for m_2. The same spring pulls back with the same magnitude, the other way:

$$m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1)$$

Step 4. Collect all coordinates on the left:

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0, \qquad m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = 0$$

Matrix form, and three free checks

$$[m]\{\ddot{x}\} + [k]\{x\} = \{F\} \qquad \text{F3.2.1}$$

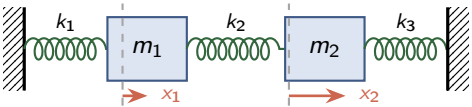
$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \qquad [k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

Three checks, before you compute anything

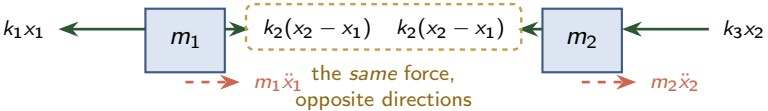
1. $[k]$ is **symmetric**: $k_{ij} = k_{ji}$. F3.2.2
2. Diagonal entries **positive**; coupling entries **negative**.
3. Each row sums to the stiffness tying that mass to *ground*. F3.2.3

An asymmetric $[k]$ is always an algebra error. Never proceed past it.

The coupling force, on both diagrams



(a) a *general* configuration, $x_2 > x_1 > 0$



(b) free-body diagram of each mass

Assembling $[k]$ by inspection

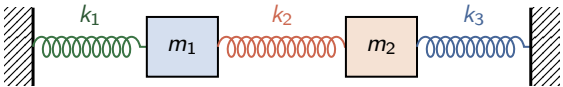
For a chain of masses you can write $[k]$ straight down, with no free bodies at all.

k_{ii}	sum of *all* stiffnesses touching mass i
k_{ij}	minus the stiffness connecting i directly to j
$k_{ij} = 0$	if i and j are not directly connected

Use inspection to CHECK the free-body result, not to replace it

The examiner marks the derivation. Write the free bodies, then confirm the matrix by inspection in one line — that is worth more than either alone.

The chain, spring by spring



k_1 touches m_1 only: adds to that diagonal.

k_3 touches m_2 only: adds to that diagonal.

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

k_2 joins them: adds to *both* diagonals, and minus itself in *both* off-diagonals.

The eigenvalue problem — where it comes from

At a natural frequency, *every* coordinate moves harmonically at the **same** frequency and in fixed proportion. So try

$$\{x\} = \{X\} \sin \omega t \qquad \Longrightarrow \qquad \{\ddot{x}\} = -\omega^2 \{X\} \sin \omega t$$

Substituting into $[m]\{\ddot{x}\} + [k]\{x\} = \{0\}$ and cancelling $\sin \omega t$:

$$\left([k] - \omega^2 [m]\right) \{X\} = \{0\}$$

Two equations, and the right-hand side is zero

$\{X\} = \{0\}$ always solves it — the system sitting still. For any *other* solution the matrix must be singular.

Why the determinant must vanish

A square system $[A]\{X\} = \{0\}$ has a non-zero solution only if $[A]$ cannot be inverted — otherwise $\{X\} = [A]^{-1}\{0\} = \{0\}$ and nothing moves.

$$\boxed{\left|[k] - \omega^2 [m]\right| = 0} \qquad \text{F3.3.1}$$

This is the **frequency equation** (or characteristic equation). For two degrees of freedom it is a *quadratic in* $\lambda = \omega^2$ — so it has two roots, and therefore two natural frequencies.

n degrees of freedom give a polynomial of degree n in λ, hence n natural frequencies. Always.

Another way to see it: two rows, one shape

Each row of $\left([k] - \omega^2 [m]\right) \{X\} = \{0\}$, on its own, *demands* a shape X_2/X_1:

$$\text{row 1: } \frac{X_2}{X_1} = \frac{k_{11} - \omega^2 m_1}{k_2}, \qquad \text{row 2: } \frac{X_2}{X_1} = \frac{k_2}{k_{22} - \omega^2 m_2}$$

At almost every ω the two rows want *different* shapes. No shape satisfies both, so the only answer is $\{X\} = \{0\}$: nothing moves.

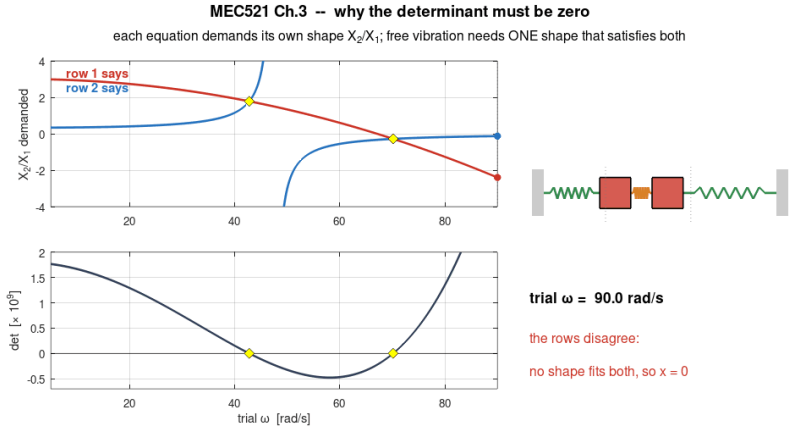
A natural frequency is where the two rows agree

Set the two ratios equal and cross-multiply:

$$\left(k_{11} - \omega^2 m_1\right)\left(k_{22} - \omega^2 m_2\right) = k_2^2$$

— which is exactly $\left|[k] - \omega^2 [m]\right| = 0$. The determinant is the agreement condition.

Watch the two rows look for agreement



The chain of the worked example that follows. The curves meet at 42.76 and 70.15 rad s^{-1} — exactly where the determinant crosses zero. Row 2 blows up at $\sqrt{k_{22}/m_2} = 47.43$ rad s^{-1}. GIF: `ch3-anim/MEC521-ch3-det.gif`.

Expanding the determinant, line by line

Two masses, $k_{21} = k_{12}$, and $\lambda = \omega^2$. A 2×2 determinant is the product down one diagonal minus the product down the other.

Step 1 — write out the determinant

$$(k_{11} - \lambda m_1)(k_{22} - \lambda m_2) - k_{12}^2 = 0$$

Step 2 — multiply out

$$m_1 m_2 \lambda^2 - m_1 k_{22} \lambda - m_2 k_{11} \lambda + k_{11} k_{22} - k_{12}^2 = 0$$

Step 3 — collect the powers of λ

$$m_1 m_2 \lambda^2 - (m_1 k_{22} + m_2 k_{11}) \lambda + \left(k_{11} k_{22} - k_{12}^2\right) = 0$$

A quadratic in λ: two roots, so two natural frequencies.

Solving the frequency equation

$$\begin{vmatrix} k_{11} - \lambda m_1 & k_{12} \\ k_{21} & k_{22} - \lambda m_2 \end{vmatrix} = 0 \qquad \lambda = \omega^2$$

$$m_1 m_2 \lambda^2 - (m_1 k_{22} + m_2 k_{11})\,\lambda + (k_{11} k_{22} - k_{12}^2) = 0$$

Three checks on the roots

1. Both roots must be **real and positive** — $\lambda = \omega^2$.
2. $\lambda_1 + \lambda_2 =$ the trace of $[m]^{-1}\,[k]$.
3. $\lambda_1 \lambda_2 =$ its determinant.

A negative or complex root means an arithmetic slip, every time.

Mode shapes: what the system does at each frequency

Put λ_1 back into *either* row of $([k] - \lambda [m])\{X\} = \{0\}$:

$$\frac{X_2}{X_1} = \frac{k_{11} - \omega_i^2 m_{11}}{-k_{12}} \qquad \text{F3.3.2}$$

Only the **ratio** is determined, never the absolute size — the right-hand side is zero, so any multiple of a solution is a solution.

So normalise, and say how

Set $X_1 = 1$ and quote X_2. The shape is $\{X\} = \{1,\ X_2/X_1\}^{\mathrm{T}}$, and the amplitude comes later from the initial conditions or the forcing.

The two modes, physically

Mode 1 — lower ω

$X_2/X_1 > 0$: both masses move the **same way** at the same instant. The coupling spring barely deforms, so the system is *soft* — hence the low frequency.

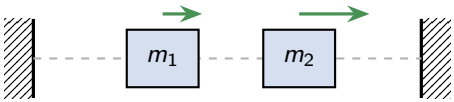
Mode 2 — higher ω

$X_2/X_1 < 0$: the masses move **in opposition**. The coupling spring is worked hard, so the system is *stiff* — hence the high frequency.

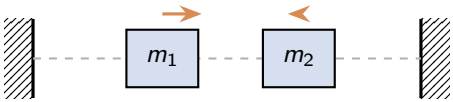
A sign check you can do without arithmetic

The lower mode is always in phase, the higher always out of phase, for a two-mass chain. If your signs come out the other way round, you have swapped λ_1 and λ_2.

The two mode shapes, drawn

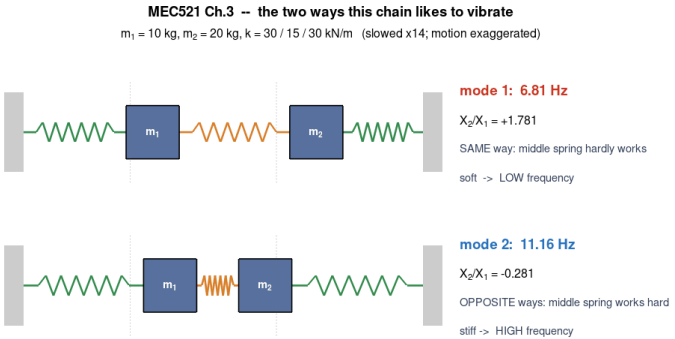


Mode 1: $\omega_1 = 42.8$ rad/s, in phase



Mode 2: $\omega_2 = 70.2$ rad/s, out of phase

Watch the two modes



The worked-example chain. Mode 1 (6.81 Hz): $X_2/X_1 = +1.781$, both the same way. Mode 2 (11.16 Hz): $X_2/X_1 = -0.281$, opposite ways — and mass 1 now moves far more than mass 2. GIF: `ch3-anim/MEC521-ch3-modes.gif`.

Worked example — frequencies and modes (1 of 3)

$m_1 = 10\,\text{kg}$, $m_2 = 20\,\text{kg}$, in a chain between two walls: $k_1 = k_3 = 30\,\text{kN m}^{-1}$, $k_2 = 15\,\text{kN m}^{-1}$ between them.

Step 1 — assemble the matrices

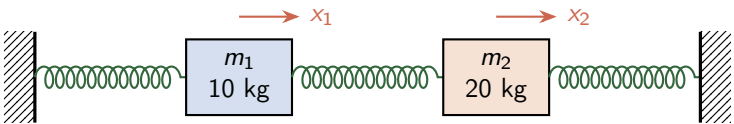
$$[m] = \begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix}, \qquad [k] = \begin{bmatrix} 45\,000 & -15\,000 \\ -15\,000 & 45\,000 \end{bmatrix}$$

$k_{11} = k_1 + k_2 = 45\,000$; $k_{22} = k_2 + k_3 = 45\,000$; $k_{12} = -k_2$.

Step 2 — run the three free checks

- symmetric ✓
- diagonal positive, off-diagonal negative ✓
- row 1 sums to $30\,000 = k_1$, the spring tying m_1 to the wall ✓

The chain we are about to solve



Worked example — frequencies and modes (2 of 3)

Step 3 — the frequency equation, $\lambda = \omega^2$:

$$\begin{vmatrix} 45\,000 - 10\lambda & -15\,000 \\ -15\,000 & 45\,000 - 20\lambda \end{vmatrix} = 0$$

$$200\lambda^2 - 1\,350\,000\lambda + 1.8 \times 10^9 = 0 \qquad \Longrightarrow \qquad \lambda^2 - 6750\lambda + 9 \times 10^6 = 0$$

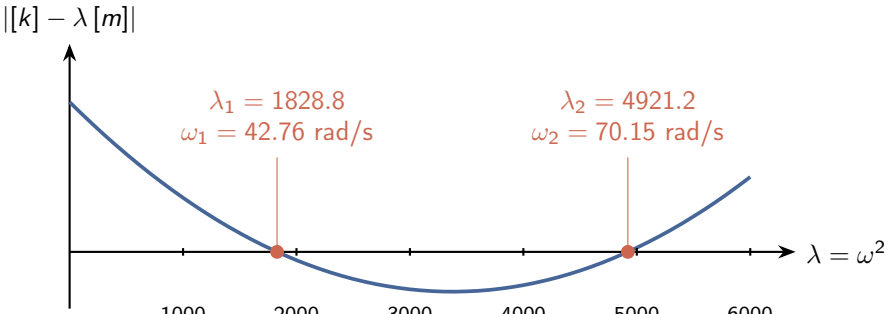
Step 4 — solve, and check against trace and determinant

$$\lambda = \frac{6750 \pm 3092.3}{2} \qquad \Longrightarrow \qquad \lambda_1 = 1828.8, \quad \lambda_2 = 4921.2$$

Sum $= 6750$ ✓ Product $= 9 \times 10^6$ ✓

$$\boxed{\omega_1 = 42.76\,\text{rad}\,\text{s}^{-1}\ (6.81\,\text{Hz}), \qquad \omega_2 = 70.15\,\text{rad}\,\text{s}^{-1}\ (11.16\,\text{Hz})}$$

The determinant crosses zero twice



only at these two values can $\{X\}$ be non-zero

Worked example — frequencies and modes (3 of 3)

Step 5 — mode shapes, from row 1

$$\frac{X_2}{X_1} = \frac{45\,000 - 10\lambda}{15\,000}$$

$$\text{Mode 1: } \frac{45\,000 - 18\,288}{15\,000} = +1.781, \qquad \text{Mode 2: } \frac{45\,000 - 49\,212}{15\,000} = -0.281$$

$$\{X\}^{(1)} = \begin{Bmatrix} 1 \\ 1.781 \end{Bmatrix}, \qquad \{X\}^{(2)} = \begin{Bmatrix} 1 \\ -0.281 \end{Bmatrix}$$

In phase at the lower frequency, out of phase at the higher — as predicted.

Step 6 — the check that catches a wrong λ Row 2 must give the same ratios: $15\,000/(45\,000 - 20\lambda)$ gives $+1.781$ and -0.281 ✓

Orthogonality

The two mode shapes are orthogonal *through the mass matrix*:

$$\{X\}^{(1)\mathrm{T}}\,[m]\,\{X\}^{(2)} = 0 \qquad \text{F3.3.3}$$

Not the ordinary dot product — $[m]$ sits in the middle. For the worked example:

$$10(1)(1) + 20(1.781)(-0.281) = 10 - 10.0 = 0 \;\checkmark$$

Why it matters

Orthogonality is what lets any motion be split into independent modal contributions. It is also the strongest single check on a computed mode shape — one line, and it catches almost every error.

Free response: superposition, and why it never repeats

Any free motion is a mixture of the two modes:

$$\{x\}(t) = A_1 \{X\}^{(1)} \cos(\omega_1 t - \phi_1) + A_2 \{X\}^{(2)} \cos(\omega_2 t - \phi_2) \qquad \text{F3.4.1}$$

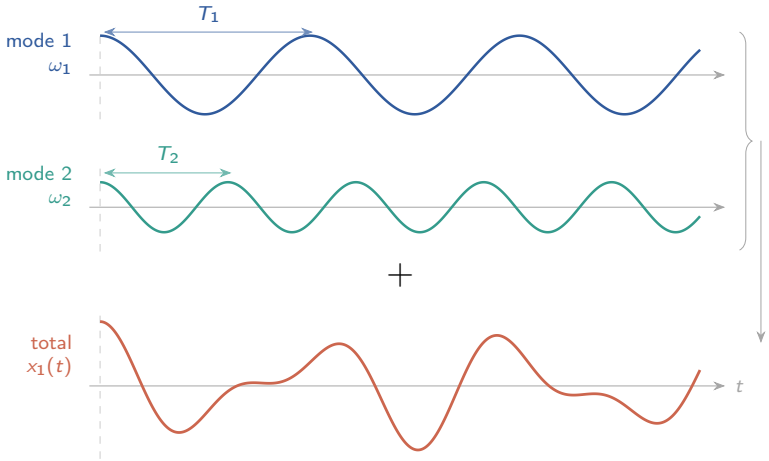
Four unknowns — A_1, A_2, ϕ_1, ϕ_2 — fixed by four initial conditions:

$x_1(0), x_2(0), \dot{x}_1(0), \dot{x}_2(0)$.

The motion is generally not periodic at all

Unless ω_2/ω_1 happens to be a ratio of whole numbers, the sum never exactly repeats. The system is perfectly deterministic and its trace still looks irregular.

Two modes, and their sum



no period — the trace never exactly repeats

Push one mass, and you get both modes

Worked-example chain again. Hold m_1 out by 10 mm, keep m_2 at rest, and let go: $\{x\}(0) = \{0.010,\ 0\}^{\mathrm{T}}$.

How much of each mode? Orthogonality picks each one out:

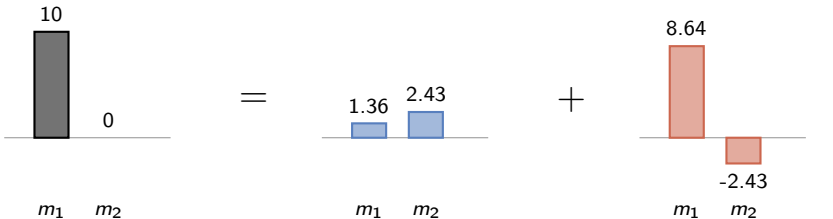
$$q_i = \frac{\{X\}^{(i)\mathrm{T}}\,[m]\,\{x\}(0)}{\{X\}^{(i)\mathrm{T}}\,[m]\,\{X\}^{(i)}} \quad\Longrightarrow\quad q_1 = \frac{0.10}{73.42} = 1.36\,\text{mm}, \qquad q_2 = \frac{0.10}{11.58} = 8.64\,\text{mm}$$

So the motion is

$$x_1 = 1.36\cos\omega_1 t + 8.64\cos\omega_2 t, \qquad x_2 = 2.43\cos\omega_1 t - 2.43\cos\omega_2 t \quad [\text{mm}]$$

Mass 1 is mostly mode 2. Mass 2 gets exactly equal amounts of each, in opposition — which is why it starts at zero. Since $\omega_2/\omega_1 = 1.640$ is not a ratio of small whole numbers, the sum never repeats.

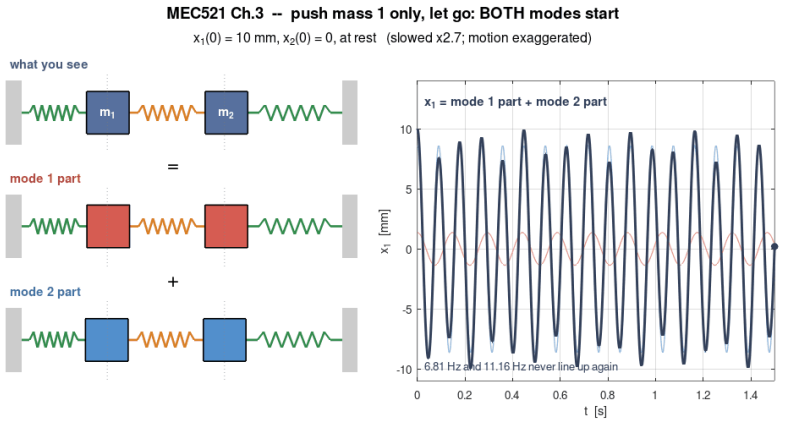
One push, two modes, drawn



$\{x\}(0)$, mm $\quad 1.36\ \{X\}^{(1)}$: mode 1 $\quad 8.64\ \{X\}^{(2)}$: mode 2

Hold m_1 out alone: both modes start, and on m_2 they cancel at $t = 0$.

Watch one push split in two



Top chain: what you would see. Below it: the two modal parts, each a perfectly regular vibration. Right: $x_1(t)$ and its two parts. GIF: `ch3-anim/MEC521-ch3-super.gif`.

Exciting one mode on its own

Displace the system *in the shape of a mode* and release it, and only that mode responds — the system vibrates as a pure sinusoid at that one frequency.

$$\{x\}(0) = \{X\}^{(1)} \qquad \Longrightarrow \qquad A_2 = 0$$

This is how modes are measured

Drive the structure at a trial frequency and adjust until every point moves in phase and passes through zero together. That is a mode, and the frequency at which it happens is a natural frequency.

Any other initial shape excites both, in proportions set by the orthogonality relation.

Beating between modes

If the two natural frequencies are close, their sum shows the same beating you met in Chapter 2 — but now with *no external forcing at all*.

$$f_{\text{beat}} = |f_2 - f_1| \qquad \text{F3.4.2}$$

Energy passes back and forth between the two masses: one grows quiet while the other grows loud, then they trade back.

A diagnostic worth knowing

A machine whose vibration slowly swells and fades, with nothing changing in the process, usually has two close modes — not a fault.

Worked example — two swings on one beam

Two playground swings, each a 25 kg child on 2.0 m chains, hang from one beam. The beam's flex couples them like a weak spring $k_c = 20\,\mathrm{N\,m^{-1}}$ between the seats. Swing 1 is pulled back 0.20 rad and released; swing 2 hangs still.

Step 1 — in phase: the beam spring is idle, each is a plain pendulum

$$\omega_1 = \sqrt{\frac{g}{L}} = \sqrt{4.905} = 2.2147\,\mathrm{rad\,s^{-1}} \quad (f_1 = 0.3525\,\mathrm{Hz})$$

Step 2 — out of phase: the spring is stretched from both ends

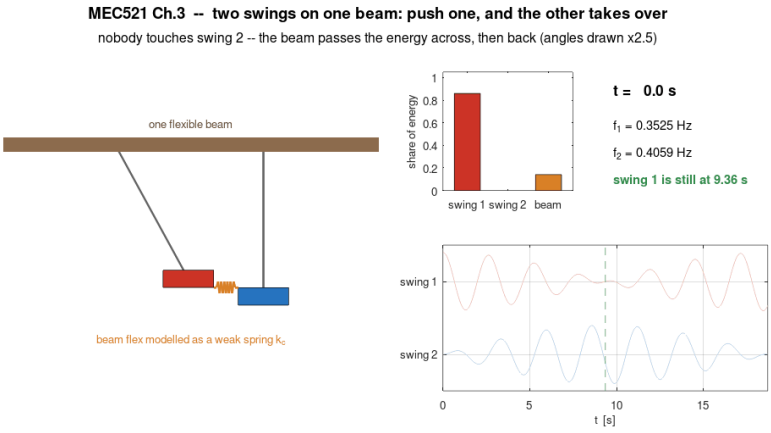
$$\omega_2 = \sqrt{\frac{g}{L} + \frac{2k_c}{m}} = \sqrt{4.905 + 1.6} = 2.5505\,\mathrm{rad\,s^{-1}} \quad (f_2 = 0.4059\,\mathrm{Hz})$$

Step 3 — the beat

$$f_{\mathrm{beat}} = 0.4059 - 0.3525 = 0.0534\,\mathrm{Hz}, \qquad T_{\mathrm{beat}} = 18.71\,\mathrm{s}$$

Half a beat later, at $t = 9.36\,\mathrm{s}$, **swing 1 is completely still** and swing 2 carries up to 99.5 % of the energy. Then it all comes back.

Watch the energy change seats



One full beat, 18.71 s, nobody touching swing 2. The bars always add up to the energy put in; the small orange bar is what the beam itself holds. GIF: `ch3-anim/MEC521-ch3-swings.gif`.

Forced two-degree-of-freedom response

Drive mass 1 harmonically and assume both masses respond at the forcing frequency:

$$\left([k] - \omega^2 [m]\right) \{X\} = \{F\}_0$$

Now the right-hand side is *not* zero, so this can simply be solved:

$$\{X\} = \left([k] - \omega^2 [m]\right)^{-1} \{F\}_0$$

Resonance is where the determinant vanishes

The inverse blows up exactly at $\omega = \omega_1$ and $\omega = \omega_2$. The frequency equation and the resonance condition are the *same* equation.

The dynamic vibration absorber

A machine is stuck at resonance and you cannot change its speed or its mount. Bolt on a *second* spring–mass, tuned to the running speed:

$$\omega^2 = \frac{k_a}{m_a} \qquad \text{F3.5.1}$$

What happens then

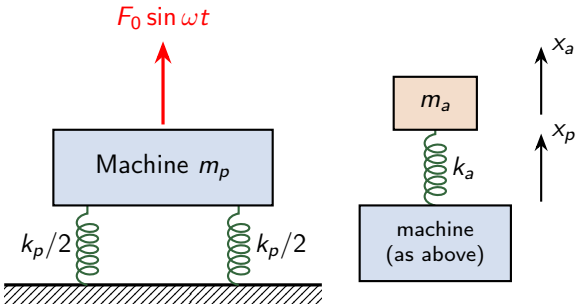
The main mass goes **still**. The absorber mass takes up all the motion and pushes back on the machine with a force exactly cancelling the excitation:

$$X_a = -\frac{F_0}{k_a} \qquad \text{F3.5.2}$$

Tuned to the *excitation* frequency — which here happens to equal the machine's natural frequency, but need not.

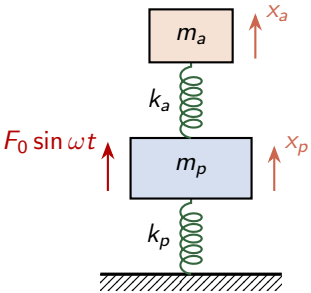
See it running: the `MEC521-Demo-Absorber` deck. GIF: `demo-absorber/MEC521-absorber-demo.gif`.

The absorber, drawn

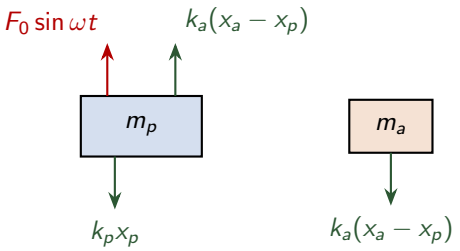


(a) machine alone (b) with absorber fitted

One free body each



(a) machine + absorber



(b) one free body each

Why the machine goes still

Both masses respond at the forcing frequency, with amplitudes X_p and X_a. The free bodies give two rows:

$$\left(k_p + k_a - \omega^2 m_p\right) X_p - k_a X_a = F_0$$

$$-k_a X_p + \left(k_a - \omega^2 m_a\right) X_a = 0$$

Step 1 — tune the absorber. Set $\omega^2 = k_a/m_a$, so $k_a - \omega^2 m_a = 0$.

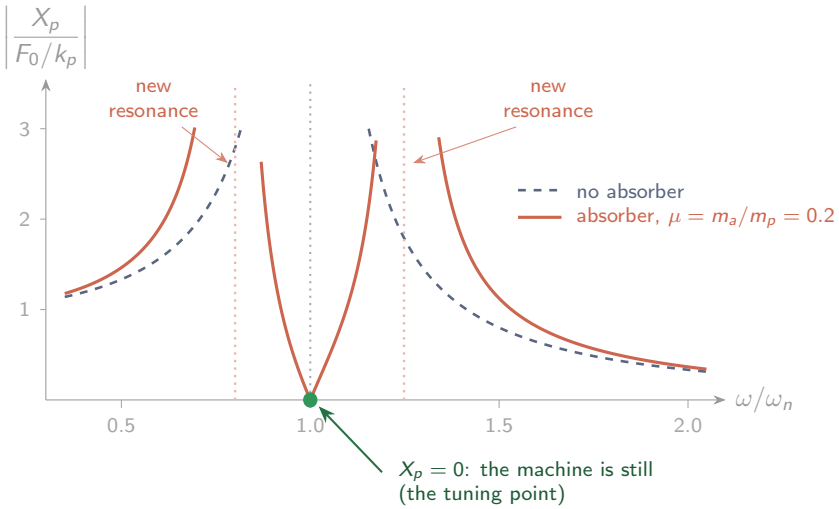
Step 2 — row 2. Only $-k_a X_p = 0$ is left, so $X_p = 0$. The machine is still.

Step 3 — row 1. With $X_p = 0$, it reads $-k_a X_a = F_0$, so

$$X_a = -\frac{F_0}{k_a}$$

The absorber's spring pushes back on the machine with exactly the excitation.

What the absorber costs



What the absorber costs

You removed one resonance and created **two**, one either side:

$$\left(\frac{\omega}{\omega_n}\right)^2 = 1 + \frac{\mu}{2} \pm \sqrt{\mu + \frac{\mu^2}{4}}, \qquad \mu = \frac{m_a}{m_p} \qquad \text{F3.5.3}$$

Small μ	new resonances close together	narrow working band
Large μ	well separated	wide band, but a heavy absorber

The absorber works at ONE speed

It is a fixed-speed cure. A machine that varies its speed will meet one of the two new resonances — which is why practical absorbers are damped.

Worked example — absorber (1 of 2)

A 200 kg machine on a mount of $k = 789.6\,\text{kN m}^{-1}$ runs at 600 rev/min — and that is exactly its natural frequency. Disturbing force 500 N. Design a 20 kg absorber.

Step 1 — confirm the diagnosis

$$\omega = \frac{2\pi(600)}{60} = 62.83\,\text{rad s}^{-1}, \qquad \omega_n = \sqrt{\frac{789\,568}{200}} = 62.83\,\text{rad s}^{-1}\ \checkmark$$

It really is at resonance — so an absorber is the right tool.

Step 2 — tune the absorber to the RUNNING speed

$$k_a = m_a\omega^2 = 20(62.83)^2 = \boxed{79.0\,\text{kN m}^{-1}}$$

Worked example — absorber (2 of 2)

Step 3 — how far does the absorber mass swing?

$$X_a = -\frac{F_0}{k_a} = -\frac{500}{78\,957} = -6.33\,\text{mm}$$

The machine is now still; the absorber swings 6.33 mm in antiphase. **Check there is room for it.**

Step 4 — the two new resonances, with $\mu = 20/200 = 0.10$:

$$\left(\frac{\omega}{\omega_n}\right)^2 = 1.05 \pm 0.3202$$

$$\omega_1 = 53.7\,\text{rad s}^{-1}\ (513\,\text{rev/min}), \qquad \omega_2 = 73.5\,\text{rad s}^{-1}\ (702\,\text{rev/min})$$

Step 5 — the engineering judgement Running speed 600 rev/min sits between them, with about 15 % margin either way. Acceptable for a fixed-speed machine; dangerous for a variable-speed one.

Reading a two-mass motion: four misreadings

1. **"The trace is irregular, so something is broken."**
   Two healthy modes at a non-whole-number ratio never repeat.
2. **"Mode 2 is the vibration of mass 2."**
   No — every mode moves *every* mass. Mode 2 here moves mass 1 the most.
3. **"Push one mass and you excite one mode."**
   You excite both, in amounts orthogonality tells you.
4. **"Beats need a forcing."**
   Two close modes beat freely — the swings had no motor.

Three ways to lose marks here

1. **An asymmetric $[k]$.**
   k_{12} must equal k_{21}. If they differ, the free bodies are wrong — stop and redo them.

2. **Quoting mode shapes as absolute amplitudes.**
   Only the *ratio* is determined. Normalise, and say what you normalised to.

3. **Tuning the absorber to ω_n instead of ω.**
   The absorber is tuned to the *excitation*. They coincide in the classic problem and not in general.

The free check that catches most of it
$\{X\}^{(1)\mathrm{T}}\,[m]\,\{X\}^{(2)} = 0$. One line, and it validates both frequencies and both shapes at once.

What the assessment will ask

Mini Project	Wk 14, 10 %	CLO3 (C5) — Ch.3–4, evaluate a real dataset.
Final	60 %	TDOF analysis is one of the topics in the CLO2 (C4) half of the paper.

What a full-mark answer looks like

- free bodies drawn, with the assumed displacement stated;
- $[m]$ and $[k]$ written out, and the symmetry check shown;
- the frequency equation formed *and* its roots checked against trace and determinant;
- mode shapes normalised, with the second row used as a check;
- a sentence saying what each mode means physically.

What to take away

1. Two degrees of freedom give **two** natural frequencies and two mode shapes — always.
2. $\left|[k] - \omega^2 [m]\right| = 0$ is the whole of the frequency calculation.
3. Mode shapes are *ratios*: lower mode in phase, higher mode out of phase.
4. An absorber tuned to ω stills the machine — and leaves two new resonances either side.

Industry alignment

Automotive, Railway Engineering, Energy — quarter-car suspension models, bogies and car-body dynamics, coupled shafts in gas turbines.

Technology integration

Data analytics essentials for modal analysis using real or simulated multivariate vibration data; correlation and orthogonality in modal data towards system diagnosis.

Lab integration

Modal analysis of an undamped TDOF system with and without a dynamic vibration absorber, using real or synthetic data.

Handbook § 3.1–3.5. Approximate methods for the fundamental frequency: Chapter 5.

The three you will reach for most: F3.2.1, F3.3.1, F3.3.2.

Everything else is on *the short list* at the front of Appendix A — one page, ordered by what you are looking at.

Tutorial: *Two-Degree-of-Freedom Systems* (two-storey shear building). Q4 releases it in a pure mode.