Forced Vibration of SDOF Systems

Resonance · imbalance · transmissibility · isolation

Abu Haqeem

MEC521 Vibrations · Handbook Chapter 2 · Weeks 4–6

Semester 20264

How to use this chapter

Three documents, three different jobs. Use the right one.

These slides	Where every formula *comes from*, and what it means. Follow the derivations here.
The handbook	*Nota padat* — the compact reference. Look up a result, a tag, a worked example. Not a first read.
The tutorial	Where you find out whether you actually understood it.

Chapter 2 has four formulae that look almost identical

They share a denominator and differ only in the numerator. Knowing *which* one a question wants is most of the skill — and most of the marks.

Where Chapter 2 is going

Chapter 1 asked what a system does when left alone. Chapter 2 asks what it does while something **keeps pushing**.

1. **A force applied directly.** Machine driven by an out-of-balance or process load. §2.1–2.2
2. **A force that grows with speed.** Rotating imbalance — the commonest excitation in industry. §2.3
3. **The support moving instead.** Vehicles, buildings, instruments. §2.4
4. **Keeping the force out.** Transmissibility and isolation design. §2.5

All four use ω_n and ζ from Chapter 1. Nothing new is assumed — only applied.

Chapter 1 vs Chapter 2

Chapter 1: what a system does after you disturb it and *leave it alone*. Chapter 2: what it does while something **keeps pushing**.

$$x(t) = \underbrace{A\sin\omega_n t + B\cos\omega_n t}_{\text{transient, at } \omega_n} + \underbrace{X\sin(\omega t - \phi)}_{\text{steady state, at } \omega}$$

Two frequencies, two jobs

The transient rings at ω_n and **dies**. The steady state runs at the **excitation** frequency ω and **persists**. Nearly every question asks only for the steady state — say so explicitly when you drop the transient. It is a marked step, not an omission.

Why forced vibration is the dangerous one

Free vibration decays. Forced vibration is fed, every cycle, for as long as the machine runs.

What resonance really does

- Amplitude limited only by damping — and damping is the parameter you know least well.
- At $\zeta = 0.02$, a 1 mm static deflection becomes 25 mm.
- Every start-up and shutdown *sweeps through* it.

Where engineers meet it

- Pump and fan baseplates.
- Washing machines on spin.
- Vehicle ride and seat isolation.
- Anything bolted to a floor that someone else designed.

Chapter 2 is the chapter you will actually use in practice.

You already know resonance: pushing a swing

At the playground

- push a child on a **buaian** once per swing, at the top, and a small push makes a big swing;
- push at a random rhythm and nothing builds — some pushes help, some fight the swing;
- the swing has its *own* rhythm. You can only feed it, not change it.

And around you

- a washing machine that "walks" as the spin speeds up, then calms;
- a car mirror or dashboard that buzzes at one speed only;
- a phone on vibrate that rattles a hollow table but not a solid one.

The whole chapter in one sentence

A small push, repeated *in time* with the system's natural rhythm, builds a large motion — unless damping drains the energy as fast as the push feeds it.

The setup: a harmonic force on an SDOF system

$$m\ddot{x} + kx = F_0 \sin \omega t \qquad \text{F2.1.1}$$

Two frequencies now appear in the same problem, and keeping them apart is the whole game:

$\omega_n = \sqrt{k/m}$	**natural**	a property of the machine
ω	**excitation**	a property of how you run it
$r = \omega/\omega_n$	**frequency ratio**	the only thing that matters

Every result in this chapter is a function of r and ζ alone.

Deriving the undamped steady-state amplitude

The forcing is harmonic at ω, so *try* a response at the same frequency: $x_p = X \sin \omega t$.

$$\ddot{x}_p = -\omega^2 X \sin \omega t$$

Substitute into $m\ddot{x} + kx = F_0 \sin \omega t$:

$$\left(-m\omega^2 + k\right) X \sin \omega t = F_0 \sin \omega t$$

$\sin \omega t$ cancels — which is exactly why the guess works:

$$X = \frac{F_0}{k - m\omega^2} = \frac{F_0/k}{1 - r^2} \qquad \text{F2.1.2}$$

Reading that result before doing any arithmetic

$$X = \frac{X_{st}}{1 - r^2}, \qquad X_{st} = \frac{F_0}{k}$$

$r \to 0$	$X \to X_{st}$	push slowly: it just deflects statically
$r < 1$	$X > 0$	moves *with* the force, in phase
$r \to 1$	$X \to \infty$	resonance — nothing limits it
$r > 1$	$X < 0$	moves *against* the force: 180° out

The negative sign is physics, not an error

Above resonance the mass cannot keep up and lags half a cycle behind. Report the magnitude, and say it is out of phase.

Resonance, and why no real machine reaches infinity

At $r = 1$ the undamped amplitude is unbounded — but that is a statement about the *model*, not about steel.

- Real systems have damping, so the peak is finite: $X = X_{st}/2\zeta$.
- Reaching a large amplitude takes *time* — it grows roughly linearly, cycle by cycle.
- Something usually yields, cracks or rubs before the steady state is ever reached.

Which is why run-up is survivable and running at resonance is not

Passing through resonance quickly is routine engineering practice. Sitting in it is a failure, every time.

Beating: what you hear when r is close to 1

Before the transient dies, two frequencies coexist — ω_n and ω — and if they are close they alternately reinforce and cancel.

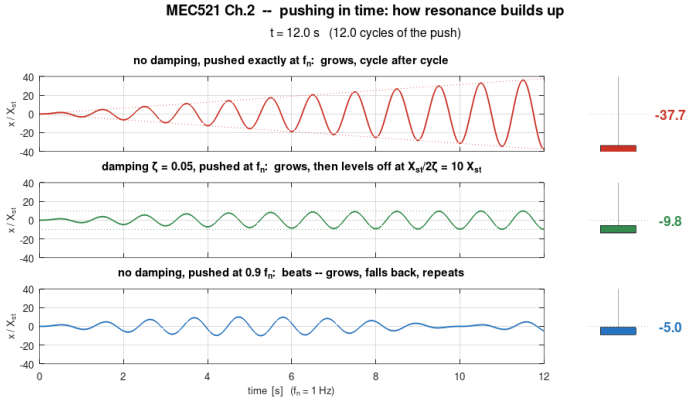
$$\sin \omega t - \sin \omega_n t = 2 \cos\left(\frac{\omega + \omega_n}{2} t\right) \sin\left(\frac{\omega - \omega_n}{2} t\right)$$

A fast oscillation at the *average* frequency, inside a slow envelope at *half the difference*.

What it sounds like on the shop floor

A throbbing, once or twice a second, from two machines running at almost the same speed. The beat rate is the speed difference — so you can tune one out by ear.

Watch resonance build up — and beat



Same push, three systems, all starting at rest ($f_n = 1\,\text{Hz}$). Undamped at f_n: $9.42X_{st}$ after 3 cycles and still growing. With $\zeta = 0.05$: levels off at $10X_{st}$. At $0.9f_n$: beats, one every 10 s, never above $10X_{st}$. GIF: `ch2-anim/MEC521-ch2-buildup.gif`.

Now add damping — and why the algebra changes

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \qquad \text{F2.2.1}$$

$\dot{x}$ brings in a $\cos \omega t$, so the response can no longer be in phase with the force. Try

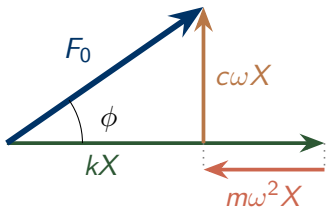
$$x_p = X \sin(\omega t - \phi)$$

Two unknowns now — an amplitude *and* a phase lag — so matching sin and cos terms gives two equations:

$$\left(k - m\omega^2\right) X = F_0 \cos \phi, \qquad c\omega X = F_0 \sin \phi$$

Square and add for X; divide for ϕ.

The four forces, drawn as arrows



At every instant four forces act on the mass. Draw each as an arrow whose length is its size and whose direction is its timing:

- spring kX and inertia $m\omega^2 X$ act along the motion, in opposite senses;
- the damper $c\omega X$ acts on the *velocity*, a quarter cycle ahead;
- the push F_0 must close the triangle.

Pythagoras on that triangle *is* the "square and add":

$$F_0^2 = \left(k - m\omega^2\right)^2 X^2 + (c\omega X)^2$$

$$\tan\phi = \frac{c\omega}{k - m\omega^2}$$

Damped harmonic response

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t$$

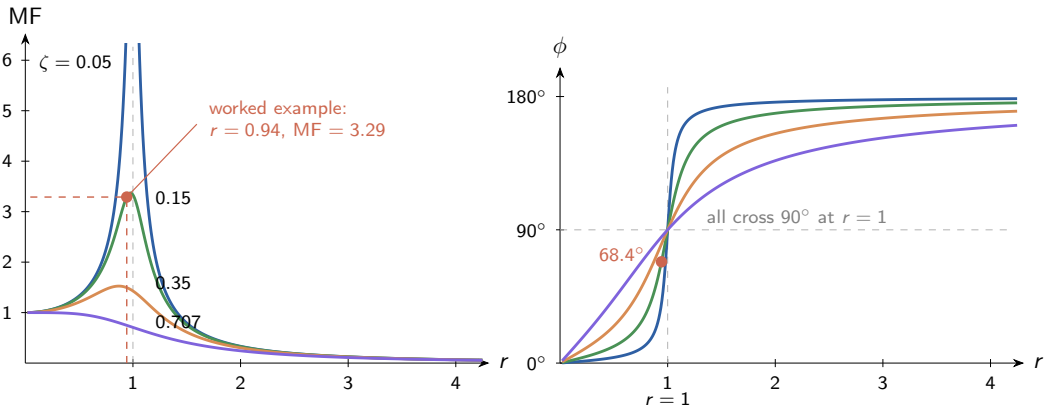
$$X = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}, \qquad \tan\phi = \frac{2\zeta r}{1-r^2}, \qquad r = \frac{\omega}{\omega_n}$$

F2.2.2 F2.2.3

$r \ll 1$	$\phi \to 0°$	stiffness-controlled
$r = 1$	$\phi = 90°$ *exactly*, whatever ζ	damping-controlled
$r \gg 1$	$\phi \to 180°$	mass-controlled

Damping matters *only* near $r = 1$.

Magnification and phase



Three regions, three different design levers

Region	Controlled by	To reduce amplitude
$r \ll 1$	stiffness k	stiffen it — damping does nothing here
$r \approx 1$	damping c	add damping, or move off resonance
$r \gg 1$	mass m	add mass — stiffening makes it *worse*

"Make it stiffer" is the reflex, and it is right only in one of the three

Identify r *before* prescribing a fix. Stiffening a mass-controlled installation pushes it towards resonance.

Where the peak actually is

Differentiate the magnification factor and set it to zero:

$$r_{\text{peak}} = \sqrt{1 - 2\zeta^2} \qquad \text{— slightly } \textit{below } r = 1$$

$$\text{MF}_{\text{max}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad \approx \frac{1}{2\zeta} \quad \text{for small } \zeta \qquad \text{F2.2.4}$$

ζ	r_{peak}	MF_{max}
0.05	0.9975	10.0
0.15	0.9772	3.37
0.30	0.9055	1.75

For light damping the distinction is academic. It stops being academic above $\zeta \approx 0.2$.

Phase is the honest resonance indicator

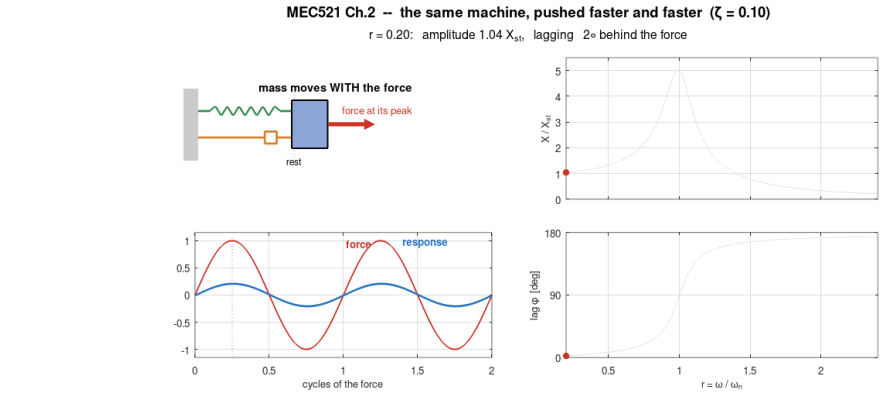
Amplitude peaks *near* $r = 1$. Phase passes 90° *exactly* at $r = 1$ — for every value of ζ.

$$\tan\phi = \frac{2\zeta r}{1 - r^2} \qquad \Longrightarrow \qquad r = 1 \;\Rightarrow\; \tan\phi = \infty \;\Rightarrow\; \phi = 90^\circ$$

Why this matters in the laboratory

The amplitude peak is broad and its position depends on damping you have not measured yet. The 90° phase crossing is sharp and damping-independent. Modal testing locates resonances by **phase**, not by amplitude.

Watch the phase lag grow



$\zeta = 0.10$. The mass is drawn at the instant the force peaks: below resonance it is pushed out with the force; at $r = 1$ it is passing through rest at full speed, 90° behind ($X = 5.00X_{st}$); far above, it is pulled the other way.

The highest point, $5.03X_{st}$, sits just below, at $r = 0.990$. GIF: `ch2-anim/MEC521-ch2-sweep.gif`.

Worked example — below resonance (1 of 2)

A 50 kg machine on a mount of $k = 20\,\text{kN m}^{-1}$, $\zeta = 0.15$, driven by $F_0 = 150\,\text{N}$ at 3 Hz.

Step 1 — get both frequencies into the same units

$$\omega_n = \sqrt{\frac{20\,000}{50}} = 20\,\text{rad s}^{-1}, \qquad \omega = 2\pi(3) = 18.85\,\text{rad s}^{-1}$$

Step 2 — the frequency ratio, and a first judgement

$$r = \frac{18.85}{20} = 0.9425$$

r is just below 1, so expect a *large* magnification and a phase lag approaching but below 90°.

Step 3 — magnification factor

$$\text{MF} = \frac{1}{\sqrt{(1 - 0.8883)^2 + (0.2827)^2}} = \frac{1}{0.3040} = 3.289$$

Worked example — below resonance (2 of 2)

Step 4 — static deflection, then scale it

$$X_{st} = \frac{F_0}{k} = \frac{150}{20\,000} = 7.5\,\text{mm} \qquad X = 3.289 \times 7.5 = \boxed{24.7\,\text{mm}}$$

Step 5 — phase

$$\tan\phi = \frac{2(0.15)(0.9425)}{1 - 0.8883} = \frac{0.2827}{0.1117} = 2.531 \qquad \phi = \boxed{68.4^\circ}$$

Here $r < 1$, so the first-quadrant value is the right one. **Step 6 — how close to the worst case?**

$$r_{\text{peak}} = \sqrt{1 - 2(0.15)^2} = 0.977, \qquad \text{MF}_{\text{max}} = 3.372$$

The machine is running just below its worst case — a 4 % speed change would make it worse, not better.

The phase quadrant trap

For $r > 1$ the phase is in the SECOND quadrant

$1 - r^2$ is negative, so $\tan\phi$ is negative — and your calculator returns the principal value, a negative angle.

$$r = 2,\ \zeta = 0.1: \qquad \tan\phi = \frac{0.4}{-3} = -0.1333 \qquad \arctan(-0.1333) = -7.6^\circ$$

$$\text{The answer is } \phi = 180^\circ - 7.6^\circ = \boxed{172.4^\circ}$$

Reporting -7.6° says the mass leads the force. It does not — above resonance it lags by nearly half a cycle.

Rotating imbalance: the commonest excitation there is

Every rotor has residual imbalance. A small mass m_0 at radius e, spinning at ω, throws a rotating force $m_0 e\omega^2$.

The crucial difference from a constant force

$$F(t) = m_0 e\,\omega^2 \sin \omega t$$

The amplitude of the forcing is **not constant** — it grows with the *square* of speed. Double the speed, quadruple the force.

Only the vertical component drives the SDOF model; the horizontal component is carried by the other degree of freedom we are choosing to ignore.

Rotating imbalance — the forcing is not constant

$$M\ddot{x} + c\dot{x} + kx = m_0 e\,\omega^2 \sin \omega t \qquad \Longrightarrow \qquad \boxed{\frac{MX}{m_0 e} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}}$$

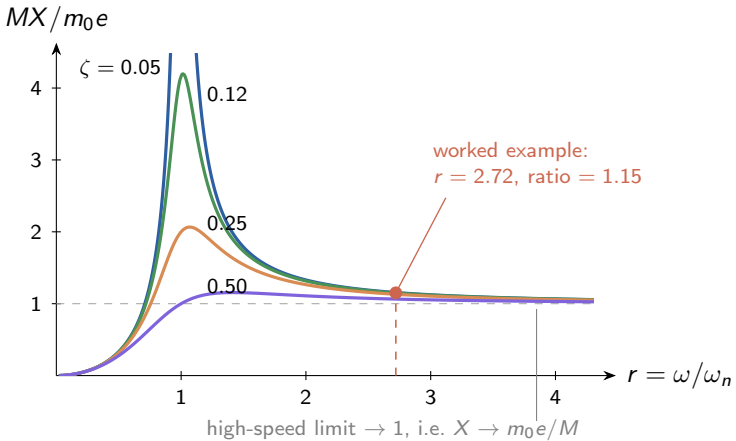
F2.3.1

	Constant force	Rotating imbalance
$r \to 0$	$X \to F_0/k$	$X \to 0$
$r \to \infty$	$X \to 0$	$X \to m_0 e/M$

M is the TOTAL mass, including m_0

Do not subtract it.

Rotating imbalance: a different curve



Why that curve has the shape it does

$$\frac{MX}{m_0 e} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

At low speed the r^2 in the numerator wins: there is barely any force, so there is barely any motion. The curve starts at **zero**. **At high speed** numerator and denominator both go as r^2 and cancel:

$$\frac{MX}{m_0 e} \to 1 \qquad \Longrightarrow \qquad X \to \frac{m_0 e}{M} \qquad \text{F2.3.2}$$

A floor you cannot design away

Running faster never reduces the amplitude below $m_0 e/M$. The only remedies are **balance the rotor** ($m_0 e$ down) or **add mass** (M up).

Worked example — unbalanced drum (1 of 2)

A washing machine, total mass 75 kg, carries 2.5 kg out of balance at 20 mm radius. Suspension $k = 90\,\text{kN m}^{-1}$, $\zeta = 0.12$. Spin speed 900 rev/min.

Step 1 — the imbalance parameter

$$m_0 e = 2.5 \times 0.020 = 0.05\,\text{kg m}$$

Step 2 — both frequencies

$$\omega_n = \sqrt{\frac{90\,000}{75}} = 34.64\,\text{rad s}^{-1}, \qquad \omega = \frac{2\pi(900)}{60} = 94.25\,\text{rad s}^{-1}$$

Step 3 — frequency ratio

$$r = \frac{94.25}{34.64} = 2.721$$

Well above resonance — so expect to be near the high-speed floor.

Worked example — unbalanced drum (2 of 2)

Step 4 — the imbalance magnification

$$\frac{MX}{m_0 e} = \frac{7.402}{\sqrt{(1-7.402)^2 + (2(0.12)(2.721))^2}} = \frac{7.402}{6.435} = 1.150$$

Step 5 — convert to an amplitude

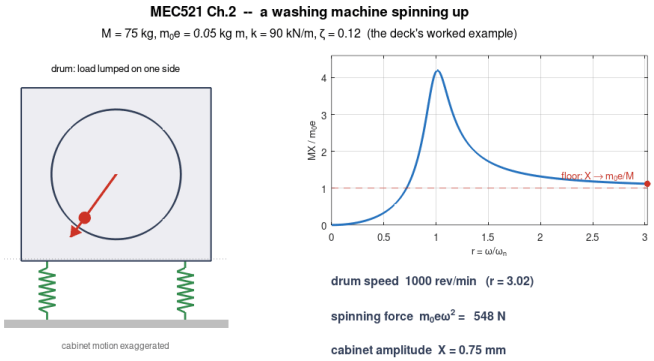
$$X = 1.150 \times \frac{m_0 e}{M} = 1.150 \times \frac{0.05}{75} = \boxed{0.767\,\text{mm}}$$

Step 6 — compare with the floor

$$\frac{m_0 e}{M} = 0.667\,\text{mm}$$

At this speed the drum is already within 15 % of the best it can ever do. **Softer suspension will not help** — only balancing will.

The unbalanced drum, spinning up



The worked example's machine run from rest to 1000 rev/min. The spinning force grows with speed squared (548 N at the top), yet the cabinet peaks at 2.78 mm while passing 331 rev/min and settles to 0.77 mm at 900 rev/min, near the 0.67 mm floor: the "walk" during spin-up. GIF: `ch2-anim/MEC521-ch2-imbalance.gif`.

Base motion: when the support moves instead

Nothing pushes the mass directly. The *ground* moves, and the mass is dragged along through the spring and damper.

Where it occurs

- a vehicle on a rough road;
- a building in an earthquake;
- an instrument on a vibrating bench;
- a seat, isolating its occupant.

What you are asked

- How much of y reaches the mass? $\rightarrow X/Y$
- How much does the spring stretch? $\rightarrow Z/Y$, with $z = x - y$

For a vehicle at speed v over a road of wavelength λ, the excitation frequency is $f = v/\lambda$ F2.4.4.

Deriving the base-motion equation

The spring and damper respond to *relative* motion, not absolute:

$$m\ddot{x} = -k(x - y) - c(\dot{x} - \dot{y})$$

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0 \qquad \text{F2.4.1}$$

With $y = Y \sin \omega t$, move the known terms to the right:

$$m\ddot{x} + c\dot{x} + kx = kY \sin \omega t + c\omega Y \cos \omega t$$

Two forcing terms, and that is the whole difference

The $c\omega Y$ term is what puts $(2\zeta r)^2$ in the **numerator** — and that single change is why base motion behaves differently at high frequency from a directly applied force.

Base motion, and three ratios that look alike

$$\frac{X}{Y} = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}} \qquad \text{F2.4.2}$$

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \qquad \text{F2.4.3}$$

Same denominator, different numerator

$$\underbrace{\frac{1}{\sqrt{\cdots}}}_{\text{MF, constant force}} \qquad \underbrace{\frac{r^2}{\sqrt{\cdots}}}_{\text{imbalance, and } Z/Y} \qquad \underbrace{\sqrt{\frac{1+(2\zeta r)^2}{\cdots}}}_{X/Y \text{ and } F_T/F_0}$$

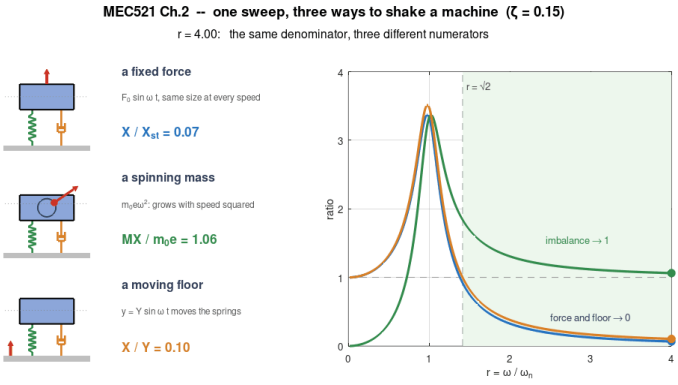
Reading the wrong one off the formula sheet is the most common Chapter 2 exam error.

How to tell which ratio a question wants

The question asks for…	**Use**
displacement under a force of fixed amplitude	$1/\sqrt{\cdots}$
amplitude due to an out-of-balance rotor	$r^2/\sqrt{\cdots}$
spring stretch, rattle space, or what a seismic pickup reads	$Z/Y = r^2/\sqrt{\cdots}$
how much ground motion reaches the mass	X/Y
how much force reaches the *floor*	$F_T/F_0 = X/Y$

Two of the five share a formula. Name the ratio you are using — examiners give credit for the identification.

Watch the three ratios part company



One sweep from $r = 0.1$ to 4 at $\zeta = 0.15$ (handbook Figure 2.2), the three machines side by side. At $r = 1$ all three peak together (3.33, 3.33 and 3.48). At $r = 4$ the fixed force gives 0.07 and the moving floor 0.10, but the spinning mass still gives 1.06: it never falls below $m_0 e/M$. GIF: `ch2-anim/MEC521-ch2-three.gif`.

Force transmissibility: what actually reaches the floor

The floor feels the spring and the damper, not the mass:

$$F_T = \sqrt{(kX)^2 + (c\omega X)^2} = kX\sqrt{1 + (2\zeta r)^2} \qquad \text{F2.5.2}$$

Divide by F_0 and substitute X:

$$\mathrm{TR} = \frac{F_T}{F_0} = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} \qquad \text{F2.5.1}$$

The same expression as X/Y — and that is not a coincidence

Force in through the mass, or motion in through the ground: the path through the mount is identical. One formula, two problems.

Isolation: the two rules

$$\mathrm{TR} = \frac{F_T}{F_0} = \frac{X}{Y} = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}$$

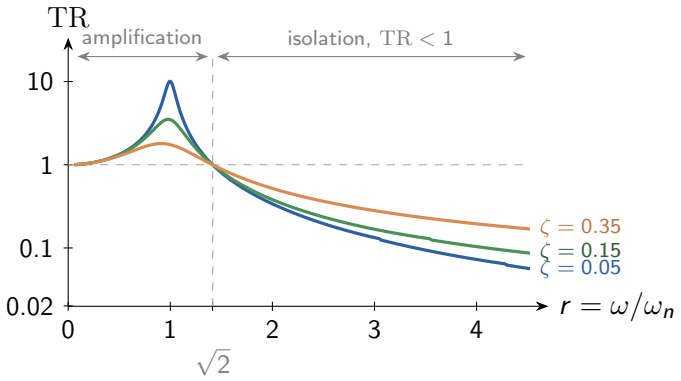
1. **Isolation requires $r > \sqrt{2}$.** Achieved by making the mount *softer*, never stiffer.
2. **Beyond $\sqrt{2}$, more damping is worse.** Damping is there to survive run-up through resonance, not to improve isolation.

Design. With $u = r^2$: $\mathrm{TR}^2 \left[(1-u)^2 + 4\zeta^2 u\right] = 1 + 4\zeta^2 u$. Take the positive root, then $\omega_n = \omega / r$, $k = m\omega_n^2$.

The required r is a minimum, so k is a MAXIMUM

Specifying a stiffer mount than the calculation allows makes the plant worse.

Transmissibility — every curve through $r = \sqrt{2}$



Why every curve passes through $r = \sqrt{2}$

Set $\mathrm{TR} = 1$ and see what survives:

$$1 + (2\zeta r)^2 = (1 - r^2)^2 + (2\zeta r)^2$$

The damping terms are *identical on both sides* and cancel:

$$1 = (1 - r^2)^2 \qquad \Longrightarrow \qquad 1 - r^2 = \pm 1 \qquad \Longrightarrow \qquad r^2 = 2$$

$$\boxed{r = \sqrt{2}} \qquad \text{— independent of } \zeta \qquad \text{F2.5.3}$$

Below $\sqrt{2}$ the mount amplifies; above it, the mount isolates

Damping cannot move that crossover. It only changes how bad things are on the way past resonance.

A corrugated road: worst at one speed, smoother faster

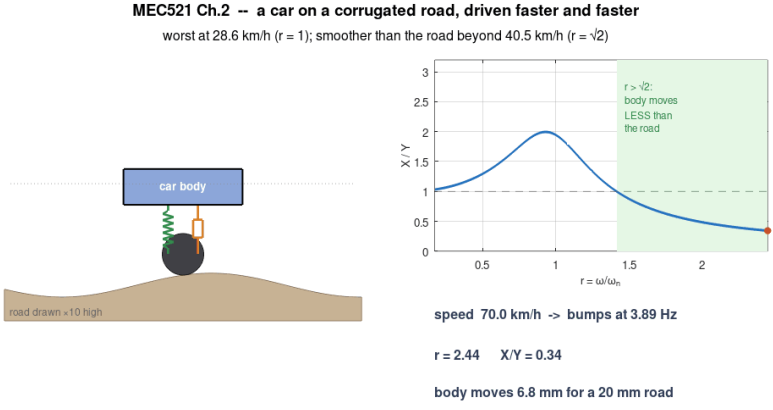
Drive along a rippled estate road and the car shakes worst at one speed; speed up and it smooths out. That is base excitation, and the chart you just saw.

A quarter car (fresh numbers): $m = 250\,\text{kg}$, $k = 25\,\text{kN m}^{-1}$, $\zeta = 0.30$, road ripples $\lambda = 5\,\text{m}$ apart and 20 mm high.

- the ripples arrive at $f = v/\lambda$; the body's own $f_n = 1.59\,\text{Hz}$;
- **worst** at $r = 1$: $v = \lambda f_n = 28.6\,\text{km h}^{-1}$, the body moving 1.99 times the road;
- **smoother than the road** beyond $r = \sqrt{2}$, i.e. $40.5\,\text{km h}^{-1}$;
- at $70\,\text{km h}^{-1}$ the body moves only 0.34 of the road: 6.8 mm.

Which is why a speed bump is meant to be crossed slowly ($r \ll 1$): crossed fast it is a hard knock, not a ripple.

Watch the car cross $r = \sqrt{2}$



Road drawn ten times higher than life. Green band: isolation, $X/Y < 1$. GIF: `ch2-anim/MEC521-ch2-road.gif`.

Designing a mount — the procedure

1. **Excitation frequency** ω from the running speed: $\omega = 2\pi N/60$.
2. **Required TR** from the specification — "no more than 15 % transmitted" means $\mathrm{TR} = 0.15$.
3. **Solve for** r from the quadratic in $u = r^2$. Take the root with $r > \sqrt{2}$.
4. $\omega_n = \omega / r$, then $k_{\max} = m\omega_n^2$.
5. **Check it is buildable**: $\delta_{st} = mg/k$. A few millimetres is practical; 50 mm is not.
6. **Check the run-up**: resonance sits at N/r rev/min. Make sure the machine can pass it quickly.

Worked example — generator set (1 of 2)

A 400 kg generator set runs at 1500 rev/min. No more than 15 % of the unbalanced force may reach the floor. Available mounts have $\zeta = 0.04$.

Step 1 — excitation frequency

$$\omega = \frac{2\pi(1500)}{60} = 157.1\ \mathrm{rad\,s^{-1}}$$

Step 2 — state the requirement as a number

$$\mathrm{TR} = 0.15, \qquad 4\zeta^2 = 0.0064$$

Step 3 — the design quadratic, with $u = r^2$ and $A = \mathrm{TR}^2 = 0.0225$:

$$0.0225u^2 - 0.05126u - 0.9775 = 0 \qquad \Longrightarrow \qquad u = 7.828, \quad r = 2.798$$

Worked example — generator set (2 of 2)

Step 4 — turn r into a stiffness

$$\omega_n = \frac{157.1}{2.798} = 56.14\,\text{rad s}^{-1}\ (8.93\,\text{Hz})$$

$$k_{\max} = m\omega_n^2 = 400(56.14)^2 = \boxed{1261\,\text{kN m}^{-1}}$$

Step 5 — is it buildable?

$$\delta_{st} = \frac{mg}{k} = \frac{400(9.81)}{1.261 \times 10^6} = 3.11\,\text{mm}$$

A 3 mm static sag is modest and entirely practical. **Step 6 — the two checks that finish the answer**

$$r = 2.80 > \sqrt{2}\ \checkmark \qquad \text{run-up resonance at } \frac{1500}{2.798} = 536\,\text{rev/min}$$

Comfortably below running speed, so the set passes through it on start-up and then leaves it behind.

Damping from a frequency-response measurement

You can read ζ straight off a measured resonance peak — no decay record needed.

$$Q = \frac{1}{2\zeta}, \qquad \Delta\omega = 2\zeta\omega_n \qquad \text{F2.5.4}$$

$\Delta\omega$ is the **half-power bandwidth** — the width of the peak where the response has fallen to $1/\sqrt{2}$ of its maximum.

Two independent routes to the same number

Chapter 1 measured ζ in the *time* domain, from a decay.
Chapter 2 measures it in the *frequency* domain, from a peak width. If they disagree, your system is not the single-degree-of-freedom system you assumed.

When a margin is not a margin

A hybrid VTOL rotor sweeps 91 Hz to 106 Hz in transition. Bracket design BPBD has a mode at 108.08 Hz, $\zeta = 0.02$ — an apparent margin of 2.08 Hz.

But how wide is that resonance?

$$Q = \frac{1}{2\zeta} = 25, \qquad \Delta f = 2\zeta f_n = 4.32\,\text{Hz}$$

Its lower half-power edge is at

$$108.08 - \zeta f_n = 108.08 - 2.16 = 105.92\,\text{Hz}$$

105.92 Hz is inside the 91 Hz to 106 Hz band

The 2 Hz margin is not a margin. A resonance is a *band*, not a line — and at low damping the band is wide.

Three beliefs the animations should have killed

1. **"At resonance the force is biggest."**
   The push is the same size at every r in the sweep. What changes is its *timing*: at $r = 1$ it lands exactly when it helps most, 90° ahead of the motion.
2. **"A stiffer mount is a safer mount."**
   Stiffer raises ω_n, lowers r, and can drag a machine from the isolation band back towards resonance.
3. **"More damping always helps."**
   Only near resonance. Beyond $r = \sqrt{2}$ the damper itself carries force into the floor.

Three ways to lose marks here

1. **Using the wrong numerator.**
   Four formulae share a denominator. Identify which one the question wants, and say so.

2. **Reporting a first-quadrant phase when $r > 1$.**
   arctan of a negative number gives a negative angle. Add 180°.

3. **Treating the required r as a target rather than a minimum.**
   $r > \sqrt{2}$ is an inequality, so k is a *maximum*. A stiffer mount fails the specification.

And one carried over from Chapter 1

N rev/min is not ω. Convert: $\omega = 2\pi N/60$.

What the assessment will ask

Assignment 1	Wk 7, 5 %	CLO2 (C4) — Ch.1–2 analysis.
Midterm	Wk 9, 20 %	CLO1 + CLO2. Ch.2 carries the analysis marks.
Final	60 %	Forced-vibration problems appear in every question.

What a full-mark answer looks like

- ω and ω_n both computed, both in $\mathrm{rad\,s^{-1}}$;
- r stated, and a sentence on what it implies before any arithmetic;
- the correct ratio named, not just used;
- a closing check — against a limit, a floor, or $\sqrt{2}$.

What to take away

1. Everything is a function of $r = \omega/\omega_n$ and ζ. Find r first, always.
2. Four formulae, one denominator. The numerator is the question.
3. Imbalance starts at zero and saturates at $m_0 e/M$ — a floor only balancing can lower.
4. Isolation needs $r > \sqrt{2}$, so mounts get *softer*, and past $\sqrt{2}$ damping hurts.

Industry alignment

Automotive, Energy, Manufacturing — suspension systems under road excitation, rotating imbalance in crank shafts, precision robotic arms.

Technology integration

IoT for smart vibration monitoring — virtual data and a dashboard for visualisation.

Lab integration

Accelerometers for real-time vibration data logging.

Handbook § 2.1–2.5.

The three you will reach for most: F2.2.2, F2.3.1, F2.5.1.
Everything else is on *the short list* at the front of Appendix A — one page, ordered by what you are looking at.

Tutorial: *Forced Vibration of SDOF Systems* (laundry extractor). One machine, the whole chapter.