Free Vibration of SDOF Systems

Modelling · natural frequency · damping

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MEC521 Vibrations · Handbook Chapter 1 · Weeks 1–3

Semester 20264

How to use this chapter

Three documents, three different jobs. Use the right one.

These slides	Where every formula *comes from*, and what it means. Follow the derivations here.
The handbook	*Nota padat* — the compact reference. Look up a result, a tag, a worked example. Not a first read.
The tutorial	Where you find out whether you actually understood it.

The tags

Boxed formulae carry a handbook tag such as **F1.3.2**. Those tags never change, even when section numbers do. Cite them in your tutorial work and your answer scripts.

Where Chapter 1 is going

By the end of week 3 you should be able to take a real machine and answer *four* questions about it.

1. **What is the model?** Reduce hardware to one mass, one spring, one damper. §1.1–1.2
2. **How fast does it want to vibrate?** Find ω_n. §1.3
3. **Does it oscillate, or just creep back?** Find ζ. §1.4
4. **How much damping does it actually have?** Measure it. §1.5

Everything in Chapter 2 assumes you can do these four

Forced vibration, resonance and isolation are all built on ω_n and ζ. If this chapter is shaky, the next one is impossible.

Why an engineer is paid to know this

Every rotating machine you will ever commission sits somewhere on a resonance curve. Your job is to know *where*.

What goes wrong

- A pump runs at the speed its baseplate happens to like.
- A cooling tower fan passes through resonance on every start-up.
- A machine is bolted down "solidly" — and transmits *more* force to the floor, not less.

What it costs

- Bearings that last months instead of years.
- Cracked welds and loosened foundation bolts.
- A machine that passes its works test and fails on site, because the site foundation is not the test-bed foundation.

None of these are material failures. They are all dynamics.

One model behind all of them

A machine on rubber mounts, a rotor overhanging a flexible shaft, and a car body on its suspension have nothing in common as hardware. They have everything in common as

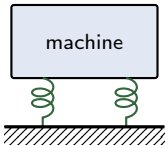
dynamics: in each, a mass is held by something elastic, and something drains the energy away.

$$m\ddot{x} + c\dot{x} + kx = 0$$

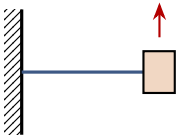
This is the whole of Chapters 1 and 2

Learn to recognise this one picture inside any machine and most of the course becomes bookkeeping.

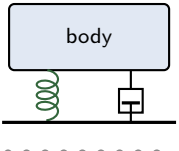
Three systems, one model



machine on
anti-vibration mounts



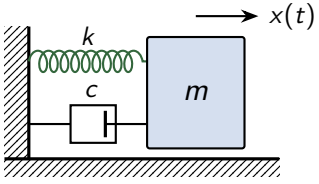
overhung rotor on
a flexible shaft



vehicle body
on its suspension



. . . are all this



$$m\ddot{x} + c\dot{x} + kx = 0$$

Why anything vibrates at all

Vibration needs energy that can be stored in *two* forms and passed back and forth.

Element	Role	Law
Mass m	stores *kinetic* energy	$f = m\ddot{x}$
Spring k	stores *potential* energy	$f = kx$
Damper c	*dissipates* energy as heat	$f = c\dot{x}$

At maximum displacement all the energy is in the spring and the mass is momentarily still. At zero displacement all of it is in the mass. The system trades one for the other, once per half-cycle.

No damper $\Rightarrow$ it never stops. No spring $\Rightarrow$ it never starts.

Vibration you have already felt

- a car hits a pothole or a *bonggol*, and the body bobs up and down a few times before it settles;
- a diving board keeps swinging after the diver has left it;
- a washing machine on spin walks across the floor;
- tall buildings in Kuala Lumpur sway when a large Sumatran earthquake strikes, hundreds of kilometres away.

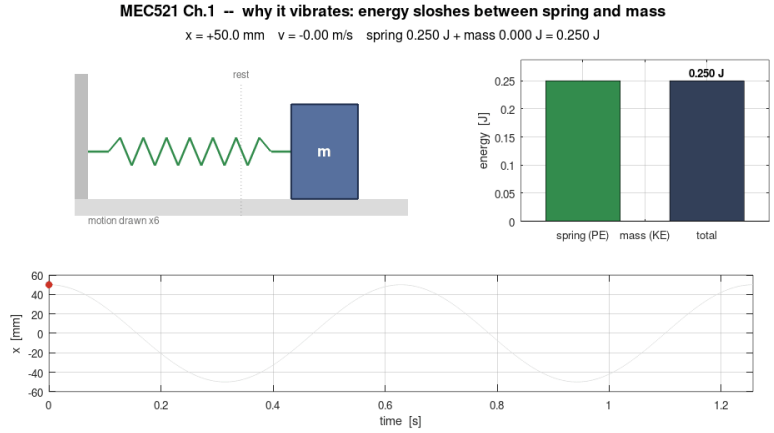
What they share

Something **springy** (tyre and coil, the board, the floor, the building's frame) and something **heavy** (the car body, the board, the drum, the floors). Push it once and it swings back and forth on its own.

The engineering question

How fast will it swing, how far, and how quickly will it stop? This whole chapter answers those three questions for the simplest case: one mass, one spring, one damper.

Watch the energy slosh



$m = 2\,\text{kg}$, $k = 200\,\text{N m}^{-1}$, pulled 50 mm and let go: $\omega_n = 10\,\text{rad s}^{-1}$. The total stays at 0.250 J — it only changes hands. GIF: `ch1-anim/MEC521-ch1-energy.gif`.

Degrees of freedom

Definition
The number of *independent coordinates* needed to specify the configuration of the system completely, at any instant.

It is a property of the **model you choose**, not of the hardware.

- An engine mount is **1 DOF** if only vertical motion matters.
- The same mount is **6 DOF** if it is not.

Nothing about the engine changed. Your question changed.

Chapters 1–2: one DOF. Chapter 3: two. Chapter 5: many.

Counting degrees of freedom — try these

1. A mass on a single spring, moving vertically.
2. A rigid bar pinned at one end, restrained by *three* springs along its length.
3. Two masses connected by a spring, free to slide.

1 DOF — one coordinate, x.

1 DOF — the bar can only rotate. One angle θ fixes every point on it, and fixes all three spring deflections with it.

2 DOF — x_1 and x_2 move independently.

Count coordinates, not components

Not the number of springs. Not the number of masses. Ask instead: how many numbers must I write down before the picture is unambiguous?

Classifying a vibration problem

Four questions, asked before any analysis. Each one changes the method.

Free or **forced**?	Free follows a disturbance and is then left alone. Forced keeps being fed energy. *Chapter 1 is free; Chapter 2 is forced.*
Damped or **undamped**?	Nothing real is undamped. The idealisation is useful when damping is small over the time you care about.
Linear or **non-linear**?	Only linear systems superpose — and superposition is what makes this whole course work.
Deterministic or **random**?	A rotating imbalance is deterministic. Road roughness and wind are not.

The vocabulary of one cycle

Every result in Chapters 1 and 2 is written in terms of simple harmonic motion:

$$x(t) = X \sin(\omega t + \phi)$$

X	amplitude	m
ω	circular frequency	$\mathrm{rad\,s^{-1}}$
ϕ	phase	rad
$T = 2\pi/\omega = 1/f$	period	s
$f = \omega/2\pi$	frequency	Hz

For a shaft turning at N rev/min, the excitation is at $\omega = 2\pi N/60$. That conversion appears in almost every forced-vibration question you will be set.

Differentiate it twice — and watch the amplitudes

$$x = X \sin(\omega t + \phi)$$

$$\dot{x} = \omega X \cos(\omega t + \phi) \qquad \text{— leads } x \text{ by } 90^\circ$$

$$\ddot{x} = -\omega^2 X \sin(\omega t + \phi) = -\omega^2 x \qquad \text{— leads } x \text{ by } 180^\circ$$

Each derivative multiplies the amplitude by ω

At 50 Hz, $\omega \approx 314\,\mathrm{rad\,s^{-1}}$. A *displacement* of 10 µm — invisible — is an *acceleration* of about 1 g. That is why machines are monitored in acceleration and isolated in displacement.

The unit trap that survives to the final answer

ω in $\mathrm{rad\,s^{-1}}$ is not f in Hz

$$\omega_n = 20\,\mathrm{rad\,s^{-1}} \qquad \Longrightarrow \qquad f_n = \frac{20}{2\pi} = 3.18\,\mathrm{Hz} \quad \textbf{not} \quad 20\,\mathrm{Hz}$$

This is the most common unit error in MEC521, and it is dangerous precisely because the wrong answer looks entirely plausible — a number, in the right range, with a unit attached.

Habit worth forming now

Write the unit on *every* line, not just the last one. A dropped $\mathrm{rad\,s^{-1}}$ is caught instantly; a wrong final number is not.

Modelling is the part that takes judgement

$$\text{real machine} \longrightarrow m,\ k,\ c \longrightarrow \text{answer}$$

The second arrow is mechanical — you will be able to do it in your sleep by week 3. The **first arrow** is engineering. It is where marks are lost, and where real designs go wrong.

What the first arrow requires

- Which motion matters, and which can be ignored?
- What is the elastic element — a spring, or the structure itself?
- Do the moving parts all move at the same speed?

Springs in combination — derive it, do not memorise it

Same displacement (parallel). Both springs stretch by x:

$$f = f_1 + f_2 = k_1 x + k_2 x = (k_1 + k_2)x \quad \Longrightarrow \quad k_{eq} = k_1 + k_2$$

Same force (series). The deflections add:

$$\delta = \delta_1 + \delta_2 = \frac{f}{k_1} + \frac{f}{k_2} \quad \Longrightarrow \quad \frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

Two lines of algebra each. If you can rebuild them you can never mix them up.

Equivalent stiffness

Parallel — same displacement, forces add

$$k_{eq} = k_1 + k_2 + \cdots$$

F1.2.1

Series — same force, deflections add

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots$$

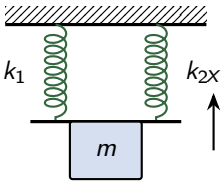
F1.2.2

Ask: *do they stretch the same amount, or carry the same force?*

The free check — one line, every time

Series is always **softer** than its softest member. Parallel is always **stiffer** than its stiffest. A k_{eq} outside that range means the wrong rule.

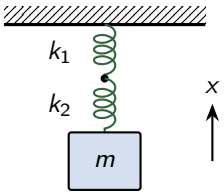
Series or parallel? Look at the deflection



Parallel

both stretch by x
forces *add*

$$k_{eq} = k_1 + k_2$$



Series

same force through both
deflections *add*

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

Worked example — equivalent stiffness (1 of 2)

A 40 kg mass hangs from a spring $k_3 = 5\,\text{kN m}^{-1}$. That spring is supported by two springs $k_1 = 8\,\text{kN m}^{-1}$ and $k_2 = 12\,\text{kN m}^{-1}$ acting side by side.

Step 1 — which pair shares what?

k_1 and k_2 span the same two points, so they must deflect *together*. Same displacement $\Rightarrow$ parallel.

$$k_{12} = k_1 + k_2 = 8000 + 12\,000 = 20\,\text{kN m}^{-1}$$

Step 2 — now what does that pair share with k_3?

Whatever force passes down through k_3 passes on into the pair. Same force $\Rightarrow$ series.

Worked example — equivalent stiffness (2 of 2)

Step 3 — combine in series

$$k_{eq} = \frac{k_{12}\,k_3}{k_{12} + k_3} = \frac{20\,000 \times 5000}{25\,000} = 4\,\mathrm{kN\,m^{-1}}$$

Step 4 — spend one line on the check

$4000 < 5000$, and $5\,\mathrm{kN\,m^{-1}}$ is the softer of the two members in parallel. A series combination *must* come out below it.

Step 5 — only now, the natural frequency

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{4000}{40}} = \boxed{10\,\mathrm{rad\,s^{-1}}} \qquad f_n = 1.59\,\mathrm{Hz}$$

Handbook Example 1.1 · every number here is asserted by `ch1_verify.py`

Elastic members are springs too

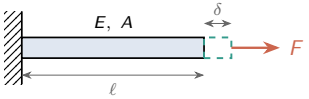
You are rarely handed a coil spring. You are handed a beam, a shaft or a bar — and it acts as a spring of stiffness $k = F/\delta$.

Member	Stiffness	Tag
Bar in tension, length ℓ, area A	$k = \dfrac{EA}{\ell}$	**F1.2.10**
Shaft in torsion	$k_t = \dfrac{GJ_p}{\ell}$	**F1.2.6**
Cantilever, load at the free end	$k = \dfrac{3EI}{\ell^3}$	**F1.2.3**
Simply supported, central load	$k = \dfrac{48EI}{\ell^3}$	**F1.2.4**
Fixed at both ends, central load	$k = \dfrac{192EI}{\ell^3}$	**F1.2.5**

These are not new formulae. A cantilever deflects $\delta = F\ell^3/3EI$; invert it.

Five members, five springs

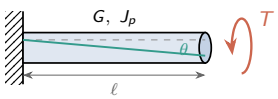
(a) Bar in tension



$$k = \frac{F}{\delta} = \frac{EA}{\ell}$$

F1.2.10

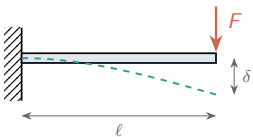
(b) Shaft in torsion



$$k_t = \frac{T}{\theta} = \frac{GJ_p}{\ell}$$

F1.2.6

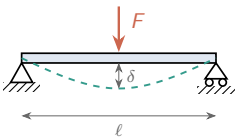
(c) Cantilever



$$k = \frac{3EI}{\ell^3}$$

F1.2.3

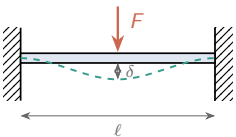
(d) Simply supported



$$k = \frac{48EI}{\ell^3}$$

F1.2.4

(e) Fixed at both ends



$$k = \frac{192EI}{\ell^3}$$

F1.2.5

Equivalent mass: equate the kinetic energy

When the moving parts do *not* all move at the same speed, m_{eq} depends on which coordinate you describe the motion with.

$$\tfrac{1}{2} m_{eq} \dot{x}_{\text{ref}}^2 = \sum \tfrac{1}{2} m_i \dot{x}_i^2 + \sum \tfrac{1}{2} J_i \dot{\theta}_i^2$$

F1.2.7

Pick the reference coordinate *first*, then express every other velocity in terms of it. The answer changes if you change the reference — and that is correct, not a mistake.

For a body rotating about a fixed pivot

$$\sum M_O = J_O \ddot{\theta}, \qquad J_O = J_G + md^2$$

with $J = \tfrac{1}{3} m\ell^2$ for a rod about its end, $\tfrac{1}{12} m\ell^2$ about its centre.

The lever arm is squared

A spring at distance a from the pivot, a damper at distance b:

$$J_O\ddot{\theta} + c\,b^2\dot{\theta} + k\,a^2\theta = 0$$

Why squared, twice over:

- the spring stretches by $a\theta$, so its force is $ka\theta$;
- that force acts at radius a, so its moment is $ka^2\theta$.

Halving the lever arm quarters the stiffness

Writing ka instead of ka^2 is one of the most common errors in Test 1, and it never looks wrong — the equation still has the right shape.

Rotation hides inside everyday machines

Something turns . . .

- a car-park **boom barrier** pivoting at its cabinet;
- a motorcycle **swingarm** pivoting behind the engine;
- a **door** on its hinges, with a closer at the top;
- a **wiper arm**; a playground **jongkang-jongket**;
- a **tyre**, a **lift drum**, a conveyor **roller**.

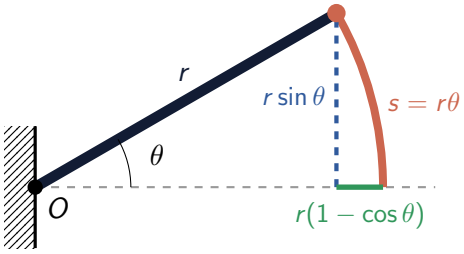
. . . and something slides

- the barrier's balance **spring** stretches;
- the rear **shock** compresses, the wheel rises;
- the closer's **piston** strokes;
- the wiper blade sweeps an **arc** of glass;
- the car, the lift car, the belt **travel**.

The engineering question

Springs and dampers only understand *linear* motion — a stretch in metres, a speed in $\mathrm{m\,s^{-1}}$. The machine only gives you an *angle*. Every equation of motion with a pivot starts by converting one into the other.

A point on a turning arm moves along an arc



1. Along the arc — exact. A radian *is* arc over radius:

$$\theta = \frac{s}{r} \quad \Longrightarrow \quad s = r\theta \qquad \text{F1.2.11}$$

2. Straight up — exact, but awkward.

$$y = r\sin\theta$$

3. Small angle. $\sin\theta \approx \theta$ and $\cos\theta \approx 1$:

$$y \approx r\theta, \qquad r(1-\cos\theta) \approx \tfrac{1}{2}r\theta^2 \approx 0$$

So at small angles, "up" and "along the arc" are the same distance, $r\theta$, and the sideways creep is second order.

How small is "small"?

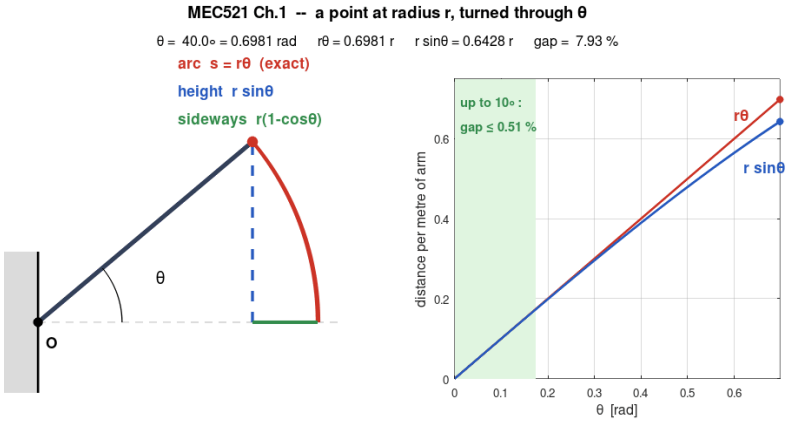
θ	θ [rad]	$\sin\theta$	gap $\dfrac{\theta - \sin\theta}{\theta}$
5°	0.0873	0.0872	0.13 %
10°	0.1745	0.1736	0.51 %
20°	0.3491	0.3420	2.02 %
30°	0.5236	0.5000	4.51 %

Vibration amplitudes are a few degrees at most, so the gap is a fraction of a percent — far below the scatter in any measured k or c.

θ must be in radians

$s = r\theta$ is only true in radians, because that is how the radian is defined. 3.4° is 0.06, not 3.4. Set the calculator to RAD before the first line.

Watch the arc and the height part company



Below 10° the red and blue lines cannot be told apart. No response from the controls? Your PDF viewer cannot animate — open `rotation-anim/MEC521-ch1-arc.gif`.

Differentiate the arc: speed and acceleration come free

The radius of a point on a rigid arm does not change, so differentiate $s = r\theta$ with r held constant:

$$s = r\theta \quad \xrightarrow{d/dt} \quad v = r\dot{\theta} \quad \xrightarrow{d/dt} \quad a_t = r\ddot{\theta}$$

who needs it	**what it needs**	**at radius r**
a spring	a stretch	$r\theta$
a damper	a speed	$r\dot{\theta}$
a mass	an acceleration (or $\frac{1}{2}mv^2$)	$r\ddot{\theta}$ (or $\frac{1}{2}mr^2\dot{\theta}^2$)

One angle, one radius each — that is all the conversion there is.

One angle, three linear motions — built one element at a time

An arm pivoted at O: spring k at radius a, damper c at b, point mass M at the tip L, the arm itself J_{arm} about O.

element	**linear motion**	**force**	**moment about O**
spring at a	stretch $a\theta$	$k\,a\theta$	$k\,a\theta \times a = ka^2\theta$
damper at b	speed $b\dot{\theta}$	$c\,b\dot{\theta}$	$c\,b\dot{\theta} \times b = cb^2\dot{\theta}$
mass at L	accel. $L\ddot{\theta}$	$ML\ddot{\theta}$	$ML\ddot{\theta} \times L = ML^2\ddot{\theta}$

Every radius appears **twice**: once converting the angle to a linear motion, once converting the force back to a moment. Hence the squares:

$$\left(J_{\text{arm}} + ML^2\right)\ddot{\theta} + cb^2\dot{\theta} + ka^2\theta = 0$$

The moment arm of a vertical force is really $a\cos\theta$; at small angles $\cos\theta \approx 1$, the same approximation as before.

Rolling without slip is the same idea

A tyre of radius R turns through θ. If it does not slip, every millimetre of rim that touches the road is laid down on it, once:

$$x = R\theta, \qquad v = R\omega, \qquad a = R\alpha \qquad \text{F1.2.11}$$

- the **contact point is momentarily at rest** — that is what "no slip" means;
- the **top of the tyre moves at** $2v$, the bottom at zero;
- with slip (wheelspin, skidding) $x \neq R\theta$ and the constraint is gone.

On the highway

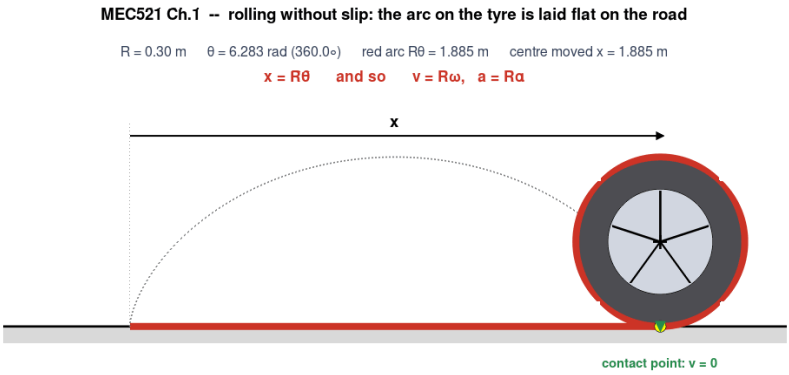
Tyre $R = 0.30\,\text{m}$ at $90\,\text{km h}^{-1}$:

$$\omega = \frac{v}{R} = \frac{25}{0.30} = 83.3\,\text{rad s}^{-1}$$

$= 796\,\text{rev/min}$. One turn moves the car $2\pi R = 1.885\,\text{m}$ — which is exactly how the odometer counts distance.

The same constraint you used in MEC524 for the cylinder rolling on a spring. A lift drum paying out rope obeys it too: rope out $= R\theta$.

Watch the rim being laid on the road



The red arc on the tyre and the red line on the road are always the same length. The yellow mark traces a cycloid and stops dead each time it touches the road. GIF: `rotation-anim/MEC521-ch1-roll.gif`.

Everyday example — why a motorbike shock is so stiff

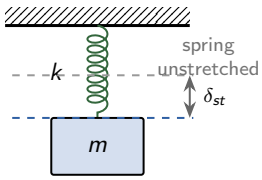
A rear swingarm pivots at O. The wheel axle is $L = 0.55\,\text{m}$ from O; the shock mounts at $b = 0.25\,\text{m}$; shock spring $k = 90\,\text{kN m}^{-1}$. What stiffness does the *wheel* feel?

Step 1 — one angle. Wheel rises $y = L\theta$; shock compresses $b\theta$.
Step 2 — moment of the shock force. $k(b\theta) \times b = kb^2\theta$.
Step 3 — as a force at the wheel. Divide the moment by L and write $\theta = y/L$:

$$F_{\text{wheel}} = \frac{kb^2\theta}{L} = k\left(\frac{b}{L}\right)^2 y \quad \Longrightarrow \quad k_{\text{wheel}} = 90\,000\left(\frac{0.25}{0.55}\right)^2 = \boxed{18.6\,\text{kN m}^{-1}}$$

A 50 mm bump strokes the shock only 22.7 mm. The lever ratio is squared, so the shock must be almost five times stiffer than the wheel rate the rider feels.

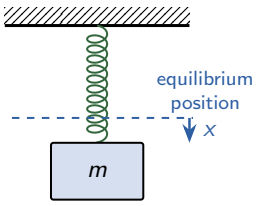
Step 1–2: set up the coordinate, then draw the diagram



(a) static equilibrium

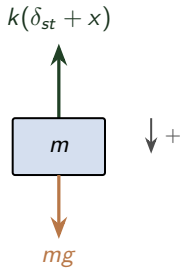
the spring already carries the weight

$k\delta_{st} = mg$



(b) displaced by x

spring now stretched by $\delta_{st} + x$



(c) free-body diagram

two forces only — take *down* as positive

Where $m\ddot{x} + kx = 0$ actually comes from

Step 1. Displace the mass by x from its *static equilibrium* position, downwards positive.

Step 2. Draw the free-body diagram. Two forces act:

- the weight mg, downwards;
- the spring force $k(\delta_{st} + x)$, upwards.

Step 3. Apply $\sum F = m\ddot{x}$, taking down as positive:

$$m\ddot{x} = mg - k(\delta_{st} + x) = mg - k\delta_{st} - kx$$

Step 4. At static equilibrium the spring already carries the weight, so $k\delta_{st} = mg$ and those two terms cancel:

$$\boxed{m\ddot{x} + kx = 0} \qquad \text{F1.3.1}$$

Why gravity disappeared — and when it does not

Measure x from static equilibrium and weight never appears

Gravity is already balanced by the spring's static deflection. It shifts the *origin*; it does not affect the motion about that origin.

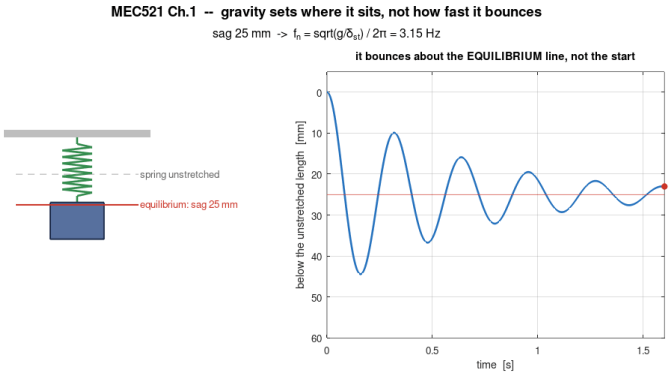
Consequences worth remembering:

- A vertical and a horizontal spring–mass system have the *same* ω_n.
- The static deflection is not wasted information — it measures the stiffness for you.

But only if x is measured from equilibrium

Measure from the unstretched spring position instead and a constant mg stays in the equation. The frequency is unchanged, but every displacement you report is offset by δ_{st}.

Watch gravity set the stage, then step aside



Hung on an unstretched spring and let go, the mass overshoots, then bounces about the line 25 mm lower — its static equilibrium — at $f_n = \sqrt{g/\delta_{st}}/2\pi = 3.15\,\text{Hz}$. Gravity moved the centre line; it did not change the bounce. GIF: `ch1-anim/MEC521-ch1-grav.gif`.

Solving it

$$m\ddot{x} + kx = 0 \qquad \Longleftrightarrow \qquad \ddot{x} + \omega_n^2 x = 0$$

Ask what function is proportional to minus its own second derivative. Sine and cosine are the only answers, so

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t = X \sin(\omega_n t + \phi)$$

$$X = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2}, \qquad \tan \phi = \frac{x_0 \omega_n}{\dot{x}_0} \qquad \text{F1.3.4}$$

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}} \qquad \text{F1.3.2}$$

What ω_n is, and what it is not

It depends on m and k. Nothing else.

Not on the initial displacement. Not on the initial velocity. Not on how hard you hit it.

Hit it harder and X gets bigger. ω_n does not move.

That is what makes natural frequency a property of the structure — and therefore something you can **design**.

The two design levers

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \text{stiffer} \Rightarrow \text{higher} \qquad \text{heavier} \Rightarrow \text{lower}$$

Both appear under a square root, so you must quadruple k to double ω_n. Stiffening is expensive.

The shortcut worth knowing on site

Since $k\delta_{st} = mg$,

$$\frac{k}{m} = \frac{g}{\delta_{st}} \qquad \Longrightarrow \qquad \boxed{\omega_n = \sqrt{\frac{g}{\delta_{st}}}} \qquad \text{F1.3.3}$$

You do not need m or k separately. **Measure how far it sags** and you have the natural frequency.

δ_{st}	ω_n	f_n
1 mm	$99\,\mathrm{rad\,s^{-1}}$	15.8 Hz
10 mm	$31\,\mathrm{rad\,s^{-1}}$	5.0 Hz
25 mm	$19.8\,\mathrm{rad\,s^{-1}}$	3.2 Hz

A softer mount means a lower f_n — which, as Chapter 2 will show, is exactly what isolation requires.

Everyday example — why a car bounces about once a second

A car body of 1200 kg settles 150 mm onto its springs when it is lowered off the jack.

Step 1 — no mass or stiffness needed, only the sag

$$\omega_n = \sqrt{\frac{g}{\delta_{st}}} = \sqrt{\frac{9.81}{0.150}} = 8.09\,\text{rad}\,\text{s}^{-1}, \qquad f_n = 1.29\,\text{Hz}, \quad T_n = 0.78\,\text{s}$$

Step 2 — load it with 300 kg **of passengers and bags.** Same springs, $k = mg/\delta_{st} = 78\,480\,\text{N}\,\text{m}^{-1}$, so the sag grows to $\delta_{st} = 1500(9.81)/78480 = 187.5\,\text{mm}$ and

$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{0.1875}} = 1.15\,\text{Hz}$$

What you feel

About one bob a second after a *bonggol* — and a fully loaded car wallows more slowly than an empty one. Heavier is slower; the formula says so, and so does your stomach.

Torsion: the same equation in disguise

Swap every translational quantity for its rotational partner and the algebra does not change at all.

displacement x	$\longrightarrow$	angle θ
mass m	$\longrightarrow$	inertia J
stiffness k	$\longrightarrow$	torsional stiffness $k_t = GJ_p/\ell$

$$J\ddot{\theta} + k_t\theta = 0 \qquad \Longrightarrow \qquad \omega_n = \sqrt{\frac{k_t}{J}} \qquad \text{F1.3.5}$$

Every result in this chapter has a torsional twin. Learn one, get both.

Worked example — torsional pendulum

Steel shaft $d = 20\,\text{mm}$, $\ell = 600\,\text{mm}$, $G = 80\,\text{GPa}$; disc $J = 0.15\,\text{kg m}^2$.

Step 1 — geometry of the shaft

$$J_p = \frac{\pi d^4}{32} = \frac{\pi (0.020)^4}{32} = 1.571 \times 10^{-8}\,\text{m}^4$$

Step 2 — turn it into a stiffness

$$k_t = \frac{G J_p}{\ell} = \frac{80 \times 10^9 \times 1.571 \times 10^{-8}}{0.600} = 2094\,\text{N m rad}^{-1}$$

Step 3 — the same formula as always

$$\omega_n = \sqrt{\frac{k_t}{J}} = \sqrt{\frac{2094}{0.15}} = \boxed{118.2\,\text{rad s}^{-1}} \qquad f_n = 18.81\,\text{Hz}$$

The factor of two that hides in plain sight

$J_p = \pi d^4/32$ for torsion, $I = \pi d^4/64$ for bending

Exactly a factor of two apart. Using the wrong one shifts ω_n by $\sqrt{2}$ — about 41 %.

That error is dangerous because it is not obviously wrong. An answer 41 % out still looks like a natural frequency.

How to keep them straight

J_p is *polar* — it resists twisting about the shaft axis, so it collects area from all around: bigger number, smaller denominator (32). I is about a diameter, so half the area contributes little: smaller number, bigger denominator (64).

Now add the damper

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad \text{F1.4.1}$$

Sine and cosine alone will no longer do — the amplitude has to decay. Try the one function that survives differentiation, $x = e^{st}$:

$$(ms^2 + cs + k)e^{st} = 0$$

e^{st} is never zero, so the bracket must be. Dividing by m:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad \Longrightarrow \qquad s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

Everything that follows is decided by one square root.

One square root decides everything

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n \qquad c_c = 2\sqrt{km} = 2m\omega_n, \qquad \zeta = \frac{c}{c_c}$$

F1.4.2 F1.4.3

$\zeta < 1$	$\zeta^2 - 1 < 0$	roots *complex* $\Rightarrow$ it oscillates
$\zeta = 1$	$\zeta^2 - 1 = 0$	roots real, equal $\Rightarrow$ the boundary
$\zeta > 1$	$\zeta^2 - 1 > 0$	roots real, distinct $\Rightarrow$ no oscillation

The sign under the square root *is* the physics. Nothing else changes.

Damping: three regimes

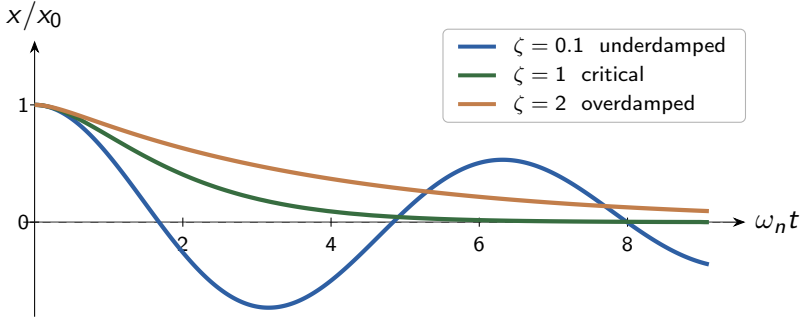
$$m\ddot{x} + c\dot{x} + kx = 0 \qquad c_c = 2\sqrt{km}, \qquad \zeta = \frac{c}{c_c}$$

$\zeta < 1$	**Underdamped**	oscillates inside a decaying envelope
$\zeta = 1$	**Critically damped**	fastest return *without* oscillating
$\zeta > 1$	**Overdamped**	returns without oscillating, but slower

More damping is not "safer"

Beyond critical, the system returns *more slowly*. Overdamped is a sluggish design, not a conservative one.

The three damping regimes, side by side



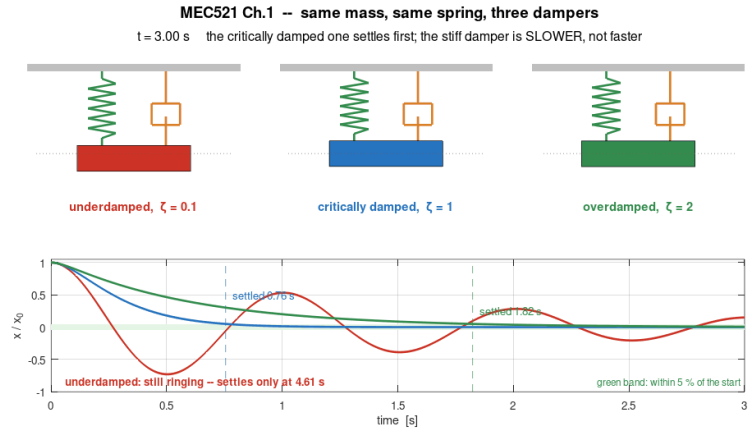
Which regime do you actually want?

device	**what it must do**	**so it is built**
door closer	shut quickly, never slam, never bounce open	close to **critical**
car shock absorber	control the bounce, yet stay soft over bumps	**underdamped**, lightly
weighing-scale needle	settle on the reading fast	close to **critical**
a guitar string	ring for as long as possible	as **little** damping as possible

More damping is not always better

Past critical, a stiffer damper makes the return *slower*: the mass creeps back. Watch the green one on the next slide.

Watch three dampers race



Same mass and spring ($f_n = 1\,\text{Hz}$), released from rest. Time to stay within 5 %: critical 0.76 s, overdamped ($\zeta = 2$) 1.82 s, underdamped ($\zeta = 0.1$) 4.61 s. GIF: `ch1-anim/MEC521-ch1-regimes.gif`.

The underdamped response — the only one that matters

Almost every engineering structure has $\zeta < 0.2$. So this is the case:

$$x(t) = e^{-\zeta\omega_n t}\left[x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right]$$

F1.4.6

$$\omega_d = \omega_n\sqrt{1-\zeta^2}, \qquad T_d = \frac{2\pi}{\omega_d}$$

F1.4.4

Read the equation as two separate things

An **oscillation** at ω_d, multiplied by an **envelope** $e^{-\zeta\omega_n t}$.

Two parameters, two different jobs

ω_d — how fast it oscillates

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

At $\zeta = 0.1$, $\omega_d = 0.995\,\omega_n$ — half a percent below ω_n.

$\zeta\omega_n$ — how fast it dies

$$\text{envelope} = e^{-\zeta\omega_n t}$$

At $\zeta = 0.1$, the amplitude is down to 5 % in about three cycles.

Light damping barely changes the frequency — but transforms the decay

This is why $\omega_d \approx \omega_n$ is a safe approximation for frequency, and a useless one for amplitude. Never approximate the decay.

Worked example — damped free response (1 of 2)

$m = 25\,\text{kg}$, $k = 10\,\text{kN m}^{-1}$, $c = 100\,\text{N s m}^{-1}$. Displaced 20 mm, released from rest. Find x at 0.30 s.

Step 1 — undamped frequency first, always

$$\omega_n = \sqrt{\frac{10\,000}{25}} = 20\,\text{rad s}^{-1}$$

Step 2 — which regime?

$$c_c = 2\sqrt{km} = 2\sqrt{10\,000 \times 25} = 1000\,\text{N s m}^{-1}, \qquad \zeta = \frac{100}{1000} = 0.10$$

$\zeta < 1$, so it is underdamped and it will oscillate.

Step 3 — the damped frequency

$$\omega_d = 20\sqrt{1 - 0.01} = 19.90\,\text{rad s}^{-1}$$

Worked example — damped free response (2 of 2)

Step 4 — put the initial conditions in. $x_0 = 0.020\,\text{m}$, $\dot{x}_0 = 0$, $\zeta\omega_n = 2.0\,\text{s}^{-1}$:

$$x(t) = e^{-2t}\left[0.020\cos 19.90t + 2.010\times 10^{-3}\sin 19.90t\right]$$

Step 5 — evaluate at $t = 0.30\,\text{s}$. Note $\omega_d t$ is in *radians*: $\omega_d t = 5.970\,\text{rad}$.

$$\cos\omega_d t = 0.9502, \qquad \sin\omega_d t = -0.3117, \qquad e^{-0.6} = 0.5488$$

$$x(0.30) = 0.5488\left[0.020(0.9502) + 2.010\times 10^{-3}(-0.3117)\right] = \boxed{10.1\,\text{mm}}$$

Set the calculator to radians

$\cos(5.970\,\text{rad}) = 0.9502$, but $\cos(5.970^\circ) = 0.9946$. Both are plausible numbers. Only one is the answer.

Damping is the one parameter you cannot calculate

Mass

From the drawing. Density times volume.

Stiffness

From the geometry. $3EI/\ell^3$, GJ_p/ℓ, a mount data sheet.

Damping

From *nowhere*. It comes from joints rubbing, material hysteresis, air, and the foundation — none of which is on the drawing.

So it is measured

Hit the structure, record the ringing, and read ζ off how quickly the amplitude falls. That is the whole of § 1.5 — and it is the only experimental method in this chapter.

Deriving the logarithmic decrement

Take two peaks n complete cycles apart. Both sit on the envelope $Xe^{-\zeta\omega_n t}$, separated in time by nT_d:

$$\frac{x_i}{x_{i+n}} = \frac{e^{-\zeta\omega_n t_i}}{e^{-\zeta\omega_n(t_i+nT_d)}} = e^{\zeta\omega_n nT_d}$$

Take logs and divide by n:

$$\delta \equiv \frac{1}{n}\ln\frac{x_i}{x_{i+n}} = \zeta\omega_n T_d$$

Substitute $T_d = 2\pi/\omega_d$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$:

$$\boxed{\delta = \frac{1}{n}\ln\frac{x_i}{x_{i+n}} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}} \qquad \text{F1.5.1}$$

Measuring damping: logarithmic decrement

$$\delta = \frac{1}{n} \ln \frac{x_0}{x_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \qquad \zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \approx \frac{\delta}{2\pi}$$

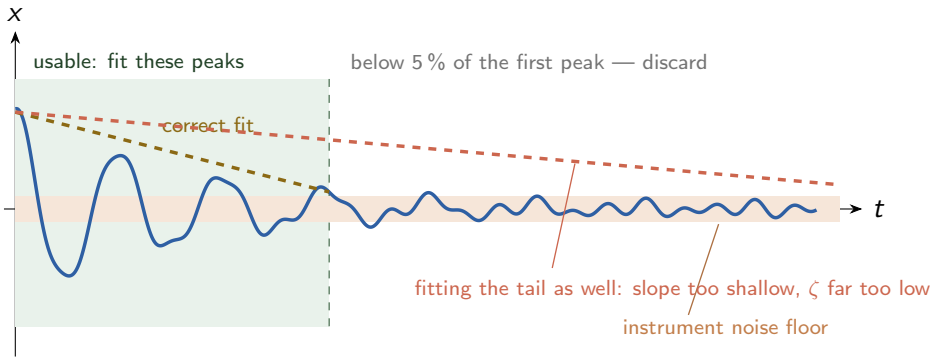
F1.5.2 F1.5.3

The light-damping approximation $\zeta \approx \delta/2\pi$ is good to better than 1 % for $\zeta < 0.2$ — which is almost everything.

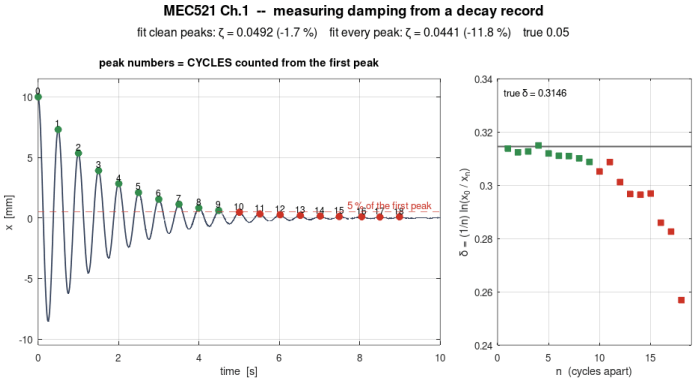
Fit the clean part of the decay only

Once the response falls into the instrument noise floor the "peaks" stop decaying. Including them flattens the slope and reports ζ several times too low. Use peaks from the largest down to about 5 % of it.

Fitting the decay: which peaks to trust



Watch the decrement being measured



A 2 Hz ring-down, true $\zeta = 0.05$ ($\delta = 0.3146$), with a 0.02 mm noise floor. Fitting the 10 peaks above 5 % gives $\zeta = 0.0492$ ($-1.7\,\%$); fitting all 19 gives $\zeta = 0.0441$ ($-11.8\,\%$). Peaks are numbered by *cycles*. GIF: `ch1-anim/MEC521-ch1-logdec.gif`.

Worked example — damping from a decay record (1 of 2)

A machine frame with $m = 12\,\text{kg}$ and $k = 4.8\,\text{kN m}^{-1}$ is struck. The amplitude falls from 8.0 mm to 2.5 mm over **5** complete cycles. Find ζ and c.

Step 1 — logarithmic decrement

$$\delta = \frac{1}{5}\ln\frac{8.0}{2.5} = \frac{1}{5}\ln 3.20 = \frac{1.1632}{5} = 0.2326$$

Step 2 — damping ratio

$$\zeta = \frac{0.2326}{\sqrt{4\pi^2 + 0.0541}} = \frac{0.2326}{6.2922} = \boxed{0.0370}$$

Step 3 — check against the approximation

$$\frac{\delta}{2\pi} = \frac{0.2326}{6.2832} = 0.0370 \quad \checkmark$$

Agreement to three figures is exactly what $\zeta \approx 0.04$ should give.

Worked example — damping from a decay record (2 of 2)

Step 4 — you measured ζ; the question wants c

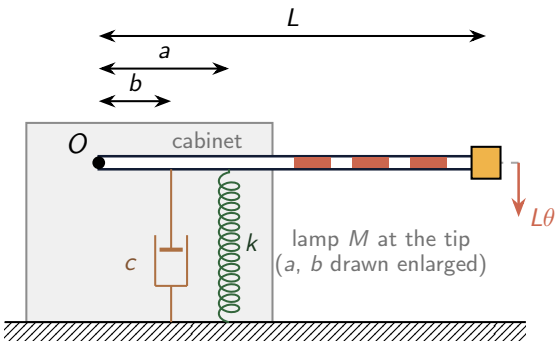
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4800}{12}} = 20\,\mathrm{rad\,s^{-1}}$$

$$c = 2\zeta m\omega_n = 2(0.0370)(12)(20) = \boxed{17.8\,\mathrm{N\,s\,m^{-1}}}$$

What just happened

Two amplitudes and a cycle count — numbers anyone can read off an oscilloscope — produced a damping coefficient in $\mathrm{N\,s\,m^{-1}}$. No model of the joints, no material data. This is how damping enters every real analysis you will do.

Worked example — the boom barrier that bounces (setup)



A car-park barrier stops at horizontal and bounces. The arm is a uniform rod, 3.0 kg, $L = 2.4$ m, pinned at O, with a 0.4 kg warning lamp at the tip. Inside the cabinet: balance spring $k = 4800\,\mathrm{N\,m^{-1}}$ at $a = 0.25$ m, damper c at $b = 0.15$ m. θ is measured from the static equilibrium, so the weight is carried by the spring pre-load and drops out.

The arm is released at rest from $\theta_0 = 0.06$ rad; the bounce falls to 30 % after **2** complete cycles. Find:

(i) the equation of motion, J_O and ω_n;

(ii) ζ and c;

(iii) the angle at $t = 1.5$ s.

Worked example — boom barrier (1 of 4: convert first)

Step 1 — is the small-angle conversion allowed? $\theta_0 = 0.06\,\text{rad} = 3.4^\circ$. The gap between $r\theta$ and $r\sin\theta$ is 0.06 %. Yes.

Step 2 — turn the one angle into three linear motions

	radius	motion	at release
lamp (tip)	$L = 2.4\,\text{m}$	$L\theta$	144 mm
spring	$a = 0.25\,\text{m}$	$a\theta$	15 mm
damper	$b = 0.15\,\text{m}$	$b\dot{\theta}$	stroke 9 mm

Step 3 — moments about O, each radius used twice

$$\underbrace{\left(\tfrac{1}{3}mL^2 + ML^2\right)}_{J_O}\ddot{\theta} + \underbrace{c\,b^2}_{\text{rot. damping}}\dot{\theta} + \underbrace{k\,a^2}_{\text{rot. stiffness}}\theta = 0$$

Worked example — boom barrier (2 of 4: inertia and ω_n)

Step 4 — inertia about the pivot, not about the centre

$$J_O = \tfrac{1}{3}(3.0)(2.4)^2 + (0.4)(2.4)^2 = 5.760 + 2.304 = \boxed{8.064\,\text{kg m}^2}$$

The lamp is 0.4 kg against the arm's 3.0 kg, yet it carries 29 % of the inertia — it sits at the largest radius, and radius is squared.

Step 5 — rotational stiffness

$$k\,a^2 = 4800\,(0.25)^2 = 300\,\text{N m rad}^{-1}$$

Step 6 — natural frequency

$$\omega_n = \sqrt{\frac{ka^2}{J_O}} = \sqrt{\frac{300}{8.064}} = \boxed{6.099\,\text{rad s}^{-1}} \qquad f_n = 0.971\,\text{Hz}$$

About one bounce a second — what you see at a car-park exit.

Step 7 — logarithmic decrement over $n = 2$ cycles

$$\delta = \frac{1}{2}\ln\frac{1}{0.30} = \frac{1.2040}{2} = 0.6020$$

Step 8 — damping ratio

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.6020}{6.3120} = \boxed{0.0954}$$

Step 9 — back from rotation to the damper

$$c\,b^2 = 2\zeta\omega_n J_O = 2(0.0954)(6.099)(8.064) = 9.382\ \mathrm{N\,m\,s\,rad^{-1}}$$

$$c = \frac{9.382}{(0.15)^2} = \boxed{417\ \mathrm{N\,s\,m^{-1}}}$$

Divide by b^2, not b. Dividing by b gives $62.5\ \mathrm{N\,s\,m^{-1}}$ — a damper nearly seven times too soft, and an answer that still looks reasonable.

Step 10 — damped frequency $\omega_d = \omega_n\sqrt{1-\zeta^2} = 6.072\,\mathrm{rad\,s^{-1}}, \quad T_d = 1.035\,\mathrm{s}$

Step 11 — the response, released from rest

$$\theta(t) = e^{-\zeta\omega_n t}\Big[\theta_0\cos\omega_d t + \frac{\zeta\omega_n\theta_0}{\omega_d}\sin\omega_d t\Big], \qquad \frac{\zeta\omega_n\theta_0}{\omega_d} = 0.005749$$

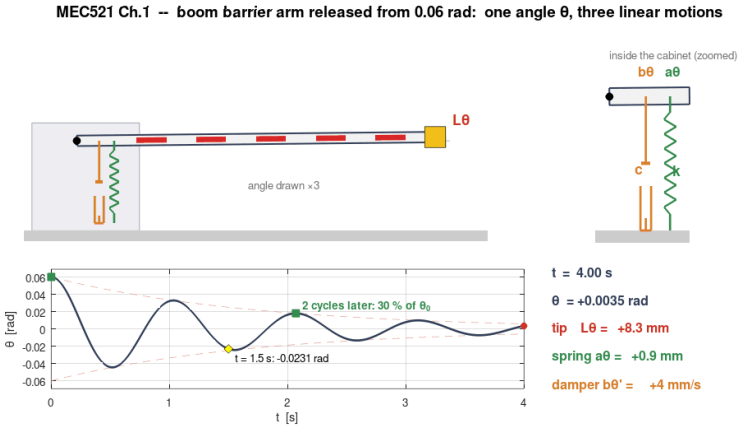
Step 12 — evaluate, calculator in RAD

$$\omega_d t = 9.1074\ \mathrm{rad}, \quad \cos = -0.9500, \quad \sin = 0.3121, \quad e^{-\zeta\omega_n t} = 0.4179$$

$$\theta(1.5) = 0.4179\,\big[0.06(-0.9500) + 0.005749(0.3121)\big] = \boxed{-0.023\,07\,\mathrm{rad}} = -1.32^\circ$$

The tip is $L\theta = -55.4\,\mathrm{mm}$ — *below* the rest line. A negative answer is not a mistake: at $t = 1.5\,\mathrm{s}$ the arm is on the far side of its swing.

The boom barrier, running



Real time. Run the same solution yourself: `Get_start_Matlab/scripts/demo1b_rotating_arm.m` prints every step above. GIF: `rotation-anim/MEC521-ch1-boom.gif`.

Converting rotation: four ways to lose marks

1. **Degrees in $s = r\theta$.**
   θ in degrees makes every length 57.3 times too big. Same for $\cos \omega_d t$ — RAD mode.
2. **ka instead of ka^2, cb instead of cb^2.**
   Each radius is used twice: once for the motion, once for the moment.
3. **J about the centre instead of the pivot.**
   $\frac{1}{12}mL^2$ is the rod about its middle. A pinned end needs $\frac{1}{3}mL^2$, and every point mass adds mr^2.
4. **$x = R\theta$ when the wheel can slip.**
   Rolling without slip is a *condition*, not a law. State it before you use it.

Three ways to lose marks here

1. **Counting peaks instead of cycles.**
   n is the number of complete *cycles* between the two amplitudes. Two successive peaks means $n = 1$, not 2.

2. **Inverting the ratio.**
   x_0 is the *earlier, larger* amplitude. Upside down gives a negative δ and a negative ζ — not a physical answer, so it is a free check.

3. **Quoting ω as f.**
   $20\,\mathrm{rad\,s^{-1}}$ is $3.18\,\mathrm{Hz}$. Write the unit on every line.

And one from earlier in the chapter

Series/parallel read off the *picture* instead of off the question. Always ask what the springs share.

What the assessment will ask

Quiz 1	Wk 4, 5 %	CLO1 — definitions, classification, ω_n, ζ from given data.
Midterm	Wk 9, 20 %	CLO1 + CLO2 — Ch.1–2. Model it, then analyse it.
Final	60 %	Every question carries CO1, CO2 and CO3.

What a full-mark answer looks like

- the model stated, with its assumptions;
- the equation of motion *derived*, not quoted;
- units on every line;
- one line of sanity check before the box.

What to take away

1. $\omega_n = \sqrt{k/m} = \sqrt{g/\delta_{st}}$ — a property of the structure, not of the disturbance.
2. Series softer, parallel stiffer. Check it every time.
3. $\zeta = c/2\sqrt{km}$ decides the regime; $\zeta < 1$ for almost everything real.
4. $\delta = \frac{1}{n}\ln(x_0/x_n)$ is how damping is actually *measured*.

Industry alignment	Technology integration	Lab integration
Automotive, Energy, Manufacturing — engine mount design, turbine shafts, pump shutdown vibration, tool chatter.	Vibration decay measurement via smartphone app (PhyPhox); data analytics for time-series signal interpretation.	MATLAB/Python simulation of vibration decay; analyse provided decay data to identify the damping type.

Handbook § 1.1–1.5.

The three you will reach for most: F1.3.2, F1.5.2, F1.2.3.
Everything else is on *the short list* at the front of Appendix A — one page, ordered by what you are looking at.

Tutorial: *Free Vibration of SDOF Systems* (anvil block). Q5 asks whether the logarithmic decrement applies at all.